AM 0219: Nonlinear Dynamical Systems

Differentiation of the solution to an initial-value problem with respect to initial time

Exercise 19: Consider a continuously differentiable vector field $f: X \times \mathbb{R} \to X = \mathbb{R}^n$. Let $x(t, t_0)$ denote the solution at time t of the associated initial-value problem

$$\dot{x}(t) = f(x(t), t), \qquad x(t_0) = x_0.$$

Prove: For any fixed τ such that $x(\tau + t_0, t_0)$ exists, there exists a neighborhood U of t_0 such that the map

$$(t_0 - \varepsilon, t_0 + \varepsilon) \to X, \qquad s \mapsto x(\tau + s, s),$$

is differentiable with respect to s, for $s \in U$. Which differential equation is solved by $v(t) := \mathrm{D}_{t_0} x(t + t_0, t_0)$?

Solution A: [Direct approach]

$$\begin{aligned} x(t) \;=\; x(t, t_0) \;&=\; x_0 + \int_{t_0}^{t} f(x(\varsigma, t_0), \varsigma)\, \mathrm{d}\varsigma \\ &=\; x_0 + \int_{t_0}^{t} f(x(\varsigma), \varsigma)\, \mathrm{d}\varsigma \qquad = \quad F(x(\,\cdot\,), t_0)(t) \end{aligned}$$

F is differentiable w.r.t. $x(\,\cdot\,)$ and t_0. Banach's Fixed-Point Theorem implies

$$w \;=\; \mathrm{D}_{t_0} x(\,\cdot\,, t_0) \;=\; \mathrm{D}_{t_0} x \;=\; \left(\mathrm{Id} - \partial_{x(\,\cdot\,)} F\Big|_{(x,t_0)}\right)^{-1} \partial_{t_0} F\Big|_{(x,t_0)},$$

and therefore

$$w \;=\; \partial_{x(\,\cdot\,)} F\Big|_{(x,t_0)} w + \partial_{t_0} F\Big|_{(x,t_0)}.$$

Thus

$$w(t) \;=\; \int_{t_0}^{t} \partial_x f(x(\varsigma), \varsigma)\, w(\varsigma)\, \mathrm{d}\varsigma - f(x(t_0), t_0)$$

and

$$\begin{aligned} v(t) \;&=\; \mathrm{D}_{t_0} x(t + t_0, t_0) \;=\; w(t + t_0) + f(x(t + t_0), t + t_0) \\ &=\; \int_{t_0}^{t+t_0} \partial_x f(x(\varsigma), \varsigma) \Big(v(\varsigma - t_0) - f(x(\varsigma), \varsigma)\Big)\, \mathrm{d}\varsigma - f(x(t_0), t_0) + f(x(t + t_0), t + t_0) \\ &=\; \int_{t_0}^{t+t_0} \partial_x f(x(\varsigma), \varsigma)\, v(\varsigma - t_0)\, \mathrm{d}\varsigma - \int_{t_0}^{t+t_0} \Big(\mathrm{D}_\varsigma f(x(\varsigma), \varsigma) - \partial_t f(x(\varsigma), \varsigma)\Big)\, \mathrm{d}\varsigma \\ &\quad\; - f(x(t_0), t_0) + f(x(t + t_0), t + t_0) \\ &=\; \int_{0}^{t} \partial_x f(x(\varsigma + t_0), \varsigma + t_0)\, v(\varsigma)\, \mathrm{d}\varsigma + \int_{0}^{t} \partial_t f(x(\varsigma + t_0), \varsigma + t_0)\, \mathrm{d}\varsigma \end{aligned}$$

and finally

$$\begin{aligned} \dot{v}(t) \;&=\; \partial_x f(x(t + t_0, t_0), t + t_0) \cdot v(t) + \partial_t f(x(t + t_0, t_0), t + t_0), \\ v(0) \;&=\; 0. \end{aligned}$$

Solution B: [Differentiation w.r.t. initial condition]

$$x(t) \quad = \quad x_0 + \int_{t_0}^{t} f(x(\varsigma),\varsigma)\,\mathrm{d}\varsigma.$$

A change of the initial time corresponds to a change of the initial value in direction of the vector field. That can be seen by inverting the flow direction:

$$x(t_0) \quad = \quad x(t) + \int_{t}^{t_0} f(x(\varsigma),\varsigma)\,\mathrm{d}\varsigma,$$

hence

$$w \quad = \quad \mathrm{D}_{t_0} x(\,\cdot\,,t_0,x_0) \quad = \quad -\mathrm{D}_{x_0} x(\,\cdot\,,t_0,x_0)\cdot f(x_0,t_0)$$

solves the variational equation

$$\dot w(t) \quad = \quad \partial_x f(x(t),t)\cdot w(t), \qquad\qquad w(t_0) \quad = \quad -f(x_0,t_0).$$

As before

$$v(t) \quad = \quad \mathrm{D}_{t_0} x(t+t_0,t_0) \quad = \quad w(t+t_0) + f(x(t+t_0),t+t_0).$$

Therefore

$$\begin{aligned}
\dot v(t-t_0) \quad &= \quad \partial_x f(x(t),t)\cdot w(t) + \partial_x f(x(t),t)\cdot \dot x(t) + \partial_t f(x(t),t)\\
&= \quad \partial_x f(x(t),t)\cdot\Big(v(t-t_0) - f(x(t),t)\Big) + \partial_x f(x(t),t)\cdot f(x(t),t) + \partial_t f(x(t),t)
\end{aligned}$$

and finally

$$\begin{aligned}
\dot v(t) \quad &= \quad \partial_x f(x(t+t_0,t_0),t+t_0)\cdot v(t) + \partial_t f(x(t+t_0,t_0),t+t_0),\\
v(0) \quad &= \quad 0.
\end{aligned}$$

Solution C: [Differentiation w.r.t. parameter]

$$\begin{aligned}
\tilde x(t) \quad = \quad x(t+t_0,t_0) \quad &= \quad x_0 + \int_0^t f(x(\varsigma+t_0,t_0),\varsigma+t_0)\,\mathrm{d}\varsigma\\
&= \quad x_0 + \int_0^t f(\tilde x(\varsigma),\varsigma+t_0)\,\mathrm{d}\varsigma \qquad\qquad = \quad F(\tilde x(\,\cdot\,),t_0)(t)
\end{aligned}$$

F is differentiable w.r.t. $\tilde x(\,\cdot\,)$ and t_0. Banach's Fixed-Point Theorem implies

$$v \quad = \quad \mathrm{D}_{t_0} x(\,\cdot\,+t_0,t_0) \quad = \quad \mathrm{D}_{t_0}\tilde x \quad = \quad \left(\mathrm{Id} - \partial_{\tilde x(\,\cdot\,)} F\Big|_{(\tilde x,t_0)}\right)^{-1} \partial_{t_0} F\Big|_{(\tilde x,t_0)},$$

and therefore

$$v \quad = \quad \partial_{\tilde x(\,\cdot\,)} F\Big|_{(\tilde x,t_0)} v + \partial_{t_0} F\Big|_{(\tilde x,t_0)}.$$

Thus

$$v(t) \quad = \quad \int_0^t \partial_x f(\tilde x(\varsigma),\varsigma+t_0)\,v(\varsigma)\,\mathrm{d}\varsigma + \int_0^t \partial_t f(\tilde x(\varsigma),\varsigma+t_0)\,\mathrm{d}\varsigma$$

and finally

$$\begin{aligned}
\dot v(t) \quad &= \quad \partial_x f(x(t+t_0,t_0),t+t_0)\cdot v(t) + \partial_t f(x(t+t_0,t_0),t+t_0),\\
v(0) \quad &= \quad 0.
\end{aligned}$$

Solution D: [Differentiation w.r.t. vector field]

$$v \;=\; \mathrm{D}_{t_0} x(\,\cdot + t_0, t_0) \;=\; \mathrm{D}_f x(\,\cdot\,, f)\, g \quad \text{with} \quad g \;=\; \partial_t f.$$

Again

$$x(t) \;=\; x(t,f) \;=\; x_0 + \int_{t_0}^{t+t_0} f(x(\varsigma, t_0), \varsigma)\,\mathrm{d}\varsigma \;=\; F(x(\,\cdot\,), f)(t)$$

F is differentiable w.r.t. $x(\,\cdot\,)$ and f. Banach's Fixed-Point Theorem implies

$$v \;=\; \mathrm{D}_f x(\,\cdot\,)\, g \;=\; \left(\mathrm{Id} - \partial_{x(\,\cdot\,)} F\Big|_{(x,f)}\right)^{-1} \partial_f F\Big|_{(x,f)}\, g,$$

and therefore

$$v \;=\; \partial_{x(\,\cdot\,)} F\Big|_{(x,t_0)} v + \partial_f F\Big|_{(x,t_0)}\, g.$$

Thus

$$v(t) \;=\; \int_{t_0}^{t+t_0} \partial_x f(x(\varsigma), \varsigma)\, v(\varsigma - t_0)\,\mathrm{d}\varsigma + \int_{t_0}^{t+t_0} g(x(\varsigma), \varsigma)\,\mathrm{d}\varsigma$$

and finally (using $g = \partial_t f$)

$$\begin{aligned} \dot v(t) &= \partial_x f(x(t+t_0, t_0), t+t_0) \cdot v(t) + \partial_t f(x(t+t_0, t_0), t+t_0), \\ v(0) &= 0. \end{aligned}$$