Exercises

AM 0219: Nonlinear Dynamical Systems

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Exercise 1: Consider a flow $\varphi(t,x) = \varphi_t(x)$ on the real axis, $x \in \mathbb{R}$.

(i) Prove or disprove: every perodic orbit is an equilibrium, i.e.

$$\forall\, x_0 \in \mathbb{R}\ \Big(\ \exists\, p > 0\ :\ \varphi(p, x_0) = x_0 \quad \Longrightarrow \quad \forall\, t \in \mathbb{R}\ :\ \varphi(t, x_0) = x_0\ \Big)$$

(ii) What are possible α-limits and ω-limits of an orbit of φ? (Consider bounded and unbounded orbits.)

Exercise 2: Consider a flow φ on $\mathbb{R}^n$, $n \geq 2$. The orbit $\varphi_t(x_0)$ of x_0 is assumed to have arbitrarily small periods, i.e.

$$\forall\, \varepsilon > 0\ \ \exists\, 0 < p < \varepsilon\ :\ \varphi_p(x_0) = x_0.$$

Prove: x_0 is an equilibrium.

Exercise 3: Let φ be a flow on $X = \mathbb{R}^n$. Sets of the form

$$\gamma(x_0)\ =\ \{\ (t, \varphi(t, x_0))\ :\ t \in \mathbb{R}\ \}\ \subset\ \mathbb{R} \times X$$

are called solution curves. Define the time shift S_ϑ on the extended phase space $\mathbb{R} \times X$ by

$$S_\vartheta\ :\quad \mathbb{R} \times X \to \mathbb{R} \times X, \qquad (t, x) \mapsto (t + \vartheta, x).$$

(i) Prove: the shift S_ϑ maps solution curves onto solution curves, for any fixed ϑ.

(ii) Which solution curves remain fixed under *a particular* S_ϑ? Which solution curves remain fixed under *all* S_ϑ?

Exercise 4: The flow $\varphi : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ of the mathematical pendulum corresponds to the vector field

$$\begin{pmatrix} \dot{x} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} v \\ -\sin x \end{pmatrix}.$$

Prove: the flow φ is *equivariant* under translations by 2π, i.e.

$$\varphi\left(t, \begin{pmatrix} x \\ v \end{pmatrix} + \begin{pmatrix} 2\pi \\ 0 \end{pmatrix}\right) = \varphi\left(t, \begin{pmatrix} x \\ v \end{pmatrix}\right) + \begin{pmatrix} 2\pi \\ 0 \end{pmatrix}, \qquad \text{for all } t, x, v.$$

Hint: You may use the uniqueness of the solution of the initial-value problem without proof.