Exercises

AM 0219: Nonlinear Dynamical Systems

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Exercise 5: The vector field $F : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

$$\dot{x} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} x,$$

with $a, b \in \mathbb{R}$. Transform this linear differential equation into polar coordinates:

$$x = \begin{pmatrix} r\cos\phi \\ r\sin\phi \end{pmatrix},$$

with $r > 0$, $\phi \in \mathbb{R}/2\pi\mathbb{Z}$. Choose $b \neq 0$ arbitrarily and sketch phase portraits in (r, ϕ)-coordinates and in x-coordinates for $a < 0$, $a = 0$, $a > 0$.

Exercise 6: The map

$$\Phi_t(x_1, x_2) = (x_1 + t, x_2 + \sigma t), \qquad \sigma \in \mathbb{R},$$

with

$$\Phi_t(x_1 + k, x_2 + n) = \Phi_t(x_1, x_2), \qquad \forall k, n \in \mathbb{Z},$$

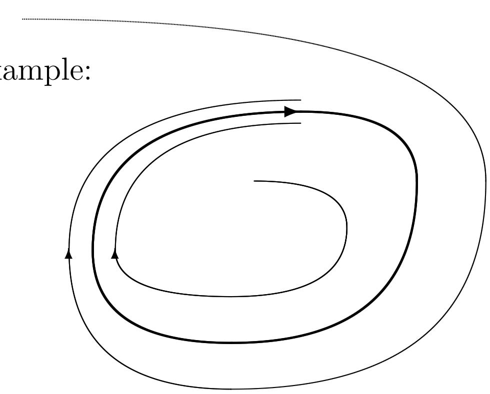
defines a flow on the 2-torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. Let σ be rational. Describe typical trajectories as well as their α- and ω-limits.

Voluntary addition: What happens for irrational σ ?

Exercise 7: Consider a periodic orbit Γ of a flow φ in $X = \mathbb{R}^n$ and a neighborhood U of Γ in X such that each trajectory $\gamma(x_0)$, $x_0 \in U$, converges to Γ as $t \to \infty$.

Prove or disprove: Every first integral of φ is constant in U.

Example:



Exercise 8: [see Arnol'd, 2.4.5] Determine all $k \in \mathbb{R}$ such that the system

$$\begin{aligned} \dot{x}_1 &= x_1 \\ \dot{x}_2 &= kx_2 \end{aligned}$$

possesses a non-constant first integral for $(x_1, x_2) \in \mathbb{R}^2$.