Exercises

AM 0219: Nonlinear Dynamical Systems

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Exercise 13: Consider the pendulum equation

$$\ddot{x} + g(x) = 0$$

for a continuous odd function g with $g(x) \cdot x > 0$ for all $x \neq 0$. Let $p(g, a) > 0$ be the minimal period of the solution to the initial value $x(0) = a > 0$, $\dot{x}(0) = 0$.

Prove:

(i) If $g_1(x) < g_2(x)$ for all $x > 0$ then $p(g_1, a) > p(g_2, a)$ for all $a > 0$.

(ii) If $x \mapsto g(x)/x$ is strictly monotonically decreasing for $x > 0$, then $a \mapsto p(g, a)$ is strictly monotonically increasing for $a > 0$.

Hint: $y(t) := \frac{a_1}{a_2} x(t)$ solves the equation $\ddot{y} + \tilde{g}(y) = 0$ with $\tilde{g}(y) := \frac{a_1}{a_2} g(\frac{a_2}{a_1} y)$.

Exercise 14: The Ricatti differential equation

$$\dot{x}(t) = x^2 + \lambda, \qquad x \in \mathbb{R}$$

depends on the parameter $\lambda \in \mathbb{R}$. Sketch the phase portraits of this dynamical system in $X = \mathbb{R}$ for $\lambda = -2$, $\lambda = -1$, and $\lambda = 1$. Which values of λ result in a similar behavior of the solutions as $\lambda = -2$? At which parameter value does that behavior change?

Exercise 15: Consider the closed, sealed-off Narragansett Bay with predator and prey fishes of total masses x and y, respectively. Suppose their dynamics obeys the Volterra-Lotka equations

$$\begin{aligned} \dot{x} &= x(\mu - \nu y), \\ \dot{y} &= y(-\varrho + \sigma x), \end{aligned}$$

with positive fixed parameters $\mu, \nu, \varrho, \sigma$. Very ($\varepsilon$-)cautious fishing would change μ into $\tilde{\mu} = \mu - \varepsilon$ and ϱ into $\tilde{\varrho} = \varrho + \varepsilon$, with $\varepsilon > 0$. Why?

Does the time-averaged prey population

$$\overline{x} := \lim_{t \to \infty} \frac{1}{t} \int_0^t x(\tau)\, \mathrm{d}\tau$$

increase or decrease, due to fishing? What happens to the total population $\overline{x + y}$?

Hint: $x = \sigma^{-1}(\dot{y}/y + \tilde{\varrho})$.

Exercise 16: Imagine a triangle of coupled "oscillators" such that each oscillator excites the next one:

$$\begin{aligned} \dot{x}_i &= f(x_i, x_{i-1}), & (i \bmod n), \qquad n = 3, \\ x(0) &:= x^0 \neq 0, & x = (x_0, \ldots, x_{n-1}) \in \mathbb{R}^n. \end{aligned}$$

Let f be smooth, $f(0,0) = 0$, and $f(0,y)y > 0$ for all $y \neq 0$. Assume that the associated flow exists globally. Whenever $x_i \neq 0$, for all i, define $z(x)$, the "zero number" of x, to be the number of sign changes of the vector x, i.e. the number of i (mod n) with $x_i x_{i-1} < 0$. Let $S(x^0)$ denote the set of times t with $x_i(t) = 0$ for at least one i. Then $z(x(t))$ is defined on the set $t \in \mathbb{R} \setminus S(x^0)$.

Prove:

(i) $S(x^0)$ is discrete;

(ii) $z(x(t_1)) \geq z(x(t_2))$, whenever $t_1 < t_2$ and $t_1, t_2 \in \mathbb{R} \setminus S(x^0)$.

Voluntary addition: Would the same conclusions hold for larger numbers $n > 3$ of oscillators?