Exercises

AM 0219: Nonlinear Dynamical Systems

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Exercise 17: Find the general solution to the differential equation

$$\dot{x}(t) = h\left(\frac{x}{t}\right),$$

by separation of variables. In particular, calculate the solution to the initial-value problem

$$\dot{x}(t) = \frac{x^2(t)}{t^2} + \frac{x(t)}{t} + 1, \qquad x(1) = x_0.$$

Hint: Derive an equation for $y(t) := x(t)/t$, as an intermediate step.

Exercise 18: Let $f : X \to X = \mathbb{R}^n$ be locally Lipschitz continuous and $J(x_0) = (t_-(x_0), t_+(x_0))$ the maximal interval of existence of the solution to the initial-value problem

$$\dot{x}(t) = f(x(t)), \qquad x(0) = x_0.$$

(i) Prove: the map $x_0 \mapsto t_+(x_0) \in (0, \infty]$ is lower semi-continuous.

(ii) Is the above map upper semi-continuous (and thus continuous)?

Remark: A map g is called lower semi-continuous if

$$\begin{array}{ll} \forall \varepsilon > 0 \;\; \exists \delta > 0 \;\; \forall x \;\; \left(|x - x_0| < \delta \Rightarrow g(x) - g(x_0) > -\varepsilon\right), & \text{in the case} \;\; g(x_0) < \infty, \\ \forall \varepsilon > 0 \;\; \exists \delta > 0 \;\; \forall x \;\; \left(|x - x_0| < \delta \Rightarrow g(x) > 1/\varepsilon\right), & \text{in the case} \;\; g(x_0) = \infty. \end{array}$$

It is called upper semi-continuous if $-g$ is lower semi-continuous.

Exercise 19: Consider a continuously differentiable vector field $f : X \times \mathbb{R} \to X = \mathbb{R}^n$. Let $x(t, t_0)$ denote the solution at time t of the associated initial-value problem

$$\dot{x}(t) = f(x(t), t), \qquad x(t_0) = x_0.$$

Prove: For any fixed τ such that $x(\tau + t_0, t_0)$ exists, there exists a neighborhood U of t_0 such that the map

$$(t_0 - \varepsilon, t_0 + \varepsilon) \to X, \qquad s \mapsto x(\tau + s, s),$$

is differentiable with respect to s, for $s \in U$. Which differential equation is solved by $v(t) := \mathrm{D}_{t_0} x(t + t_0, t_0)$?

Exercise 20: The initial-value problem

$$\dot{x} = f(x) = x^2, \qquad x(0) = 1$$

has a solution for $-\infty < t < 1$ with "blow-up", $\lim_{t\to 1} x(t) = +\infty$.

Let $(x_k)_{k\in\mathbb{N}}$ be the series of Picard iterates:

$$\begin{aligned} x_0(t) &\equiv 1, \\ x_{n+1}(t) &= 1 + \int_0^t f(x_n(s))\,\mathrm{d}s. \end{aligned}$$

(i) Prove: $x_k(t)$ is defined for all $k \in \mathbb{N}$ and $t \in \mathbb{R}$.

(ii) Calculate $x_1(t), x_2(t), x_3(t)$ and $x_4(t)$ explicitly.

(iii) Determine all $t \in \mathbb{R}$ such that $x_k(t)$ converges to the solution $x(t)$ of the initial-value problem, as $k \to \infty$.