Exercises

AM 0219: Nonlinear Dynamical Systems

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Exercise 25: Calculate the solutions of the following linear differential equations

$$\text{(i)} \qquad \dot{x} = \begin{pmatrix} -3 & 0 & 2 \\ -1 & -3 & 5 \\ -1 & 0 & 0 \end{pmatrix} x, \qquad x(0) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\text{(ii)} \qquad \dot{x} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{pmatrix} x, \qquad x(0) = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}.$$

Exercise 26: We want to understand the damped linear pendulum

$$\ddot{x} + \nu\dot{x} + \omega^2 x = 0$$

with parameters $\boldsymbol{\nu, \omega > 0}$ and initial conditions $x(0) = 0$, $\dot{x}(0) = 1$.

(i) Determine the explicit solution of the given initial-value problem for all ν, ω. Sketch phase portraits and a diagram of the (ν, ω)-plane of different qualitative behavior.

(ii) How does the phase portrait change at the boundary between different zones in the (ν, ω)-plane, for instance due to a change of the damping? How do you reconcile the discontinuities in the phase portraits of the Jordan normal form with differentiable dependence of the flow on the parameters ν, ω?

Exercise 27: Consider the linear system

$$\begin{aligned} \dot{x} &= Ax + D(y - x), \\ \dot{y} &= Ay + D(x - y), \end{aligned}$$

with $x, y \in \mathbb{R}^n$, which models two symmetrically coupled oscillators. D denotes a diagonal matrix, $D = \operatorname{diag}(d_1, \ldots, d_n)$, with strictly positive entries, $d_i > 0$. Furthermore, let $\Re\mathrm{e}\,\operatorname{spec}(A) < 0$.

(i) Prove: If $x(0) = y(0)$ then $x(t), y(t) \longrightarrow 0$ as $t \longrightarrow +\infty$.

(ii) Find matrices A, D (with the above constraints) such that $x(t) \longrightarrow \infty$ as $t \longrightarrow +\infty$ for some initial condition $x(0)$, $y(0)$.

Hint: In the second part, choose $n = 2$ and consider the invariant subspace $\{x = -y\}$.

Exercise 28: [LISSAJOUS figures] Let A be a symmetric real (2×2)-matrix

$$A = \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}.$$

Consider the Hamilton system with Hamilton function $H(x, \dot{x}) = \frac{1}{2}(\dot{x}^{\mathrm{T}}\dot{x} + x^{\mathrm{T}}Ax)$:

$$(*) \qquad \ddot{x} = -Ax.$$

(i) Transform $(*)$ into a system of decoupled pendulum equations (ω_1, ω_2 real):

$$(**) \qquad \left\{ \begin{array}{rcl} \ddot{y}_1 + \omega_1^2 y_1 & = & 0, \\ \ddot{y}_2 + \omega_2^2 y_2 & = & 0, \end{array} \right.$$

(ii) Sketch the solution $(x_1(t),\ x_2(t))$ of $(*)$ for

$$A = \begin{pmatrix} 5 & -4 \\ -4 & 5 \end{pmatrix}$$

with initial conditions $x_1 = x_2 = \dot{x}_1 = -\dot{x}_2 = 1$.