Exercises

AM 0219: Nonlinear Dynamical Systems

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due date: Mon, Nov 08, 2004

Exercise 29: We want to count the number k_n of (pairs of) FIBONACCIs rabbits after n months. There are two hatchlings each month for each pair of rabbits which are at least two months old.

$$k_{n+1} \;=\; k_n + k_{n-1}, \qquad k_0 = k_1 = 1.$$

(i) Write the above two-term recursion as a linear iteration $x_{n+1} = Ax_n$ with a suitably chosen vector x and matrix A.

(ii) Transform A into JORDAN normal form and calculate an explicit formula for k_n.

(iii) Prove that the ratios

$$r_n \;=\; k_{n+1}/k_n$$

converge to the "golden ratio".

Remark: Look for the "golden ratio" and the FIBONACCI sequence in architecture (Acropolis of Athens, Duomo of Florence), music (Bach, Mozart, Bartok), arts (da Vinci, Michelangelo, Dali), esthetics (Is the golden ratio the most beautiful proportion?), nature (conifers, sunflower), and many other aspects of life. Be aware, however, that some of the a-posteriori discoveries of the golden ratio in nature, architecture, and arts are based on questionable measurements, misinterpretation of nearby rational proportions (e.g. 5:3), and misuse of statistical methods.

Exercise 30: [Variation-of-constants formula for maps] Consider a Banach space X, a linear map $A : X \to X$, and a nonlinear map $f : \mathbb{N} \to X$.

Complete and prove the formula

$$x(n) \;=\; A^{\bigcirc} x_0 + \sum_{k=\bigcirc}^{\bigcirc} A^{\bigcirc} f(\bigcirc),$$

for the unique solution $x : \mathbb{N} \to X$ to the initial-value problem

$$x(0) \;=\; x_0, \qquad x(n+1) \;=\; A\,x(n) + f(n), \qquad \forall n \in \mathbb{N}.$$

Exercise 31: Let $I \subset \mathbb{R}$ be an interval and $A \in C^1(I, \mathbb{R}^{n\times n})$.
Prove: If A and $\dot{A}$ commute, i.e. if $[A(t), \dot{A}(t)] := A(t)\dot{A}(t) - \dot{A}(t)A(t) = 0$ for all $t \in I$, then

$$\frac{\mathrm{d}}{\mathrm{d}t} e^{A(t)} \;=\; \dot{A}(t)\, e^{A(t)} \;=\; e^{A(t)}\, \dot{A}(t).$$

Exercise 32: Consider two linear dynamical systems in the plane,

$$(1) \qquad \dot{x} \;=\; Ax, \qquad\qquad (2) \qquad \dot{x} \;=\; Bx,$$

$x = (x_1, x_2) \in \mathbb{R}^2$, $A = (a_{ij})_{1\le i,j\le 2}$, $B = (b_{ij})_{1\le i,j\le 2}$. Combine both system to the piecewise linear system

$$(3) \qquad \dot{x} \;=\; \begin{cases} Ax & \text{for} \quad x_1x_2 > 0 \\ Bx & \text{for} \quad x_1x_2 < 0 \end{cases}$$

We call a function $x : (0,T) \to \mathbb{R}$ a solution of (3) to the initial condition $x(0)$ if

- x is Lipschitz-continuous and piecewise differentiable.
- x solves (3) at each point $x(t)$ of the set

$$\Omega \;=\; \{\, (x_1, x_2) \in \mathbb{R}^2 \,|\, x_1x_2 \neq 0 \,\} \cup \{\, (0,0) \,\}$$

- x intersects the complement of Ω only at discrete times, i.e. the set $\{t \in I \,|\, x(t) \notin \Omega\}$ is discrete in $\mathbb{R}$.

(i) Prove: If $a_{12}b_{12} > 0$ and $a_{21}b_{21} > 0$ then system (3) has a global and unique solution for any initial condition.

(ii) Prove or disprove: If, under the assumptions of (i), the origin is asymptotically stable in (1) as well as (2), i.e. if $\Re\mathrm{e}\,\mathrm{spec}\, A < 0$ and $\Re\mathrm{e}\,\mathrm{spec}\, B < 0$, then the origin is asymptotically stable in (3).