Exercises

AM 0219: Nonlinear Dynamical Systems

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Exercise 37: Let $A \subseteq B \subseteq X = \mathbb{R}^N$ be sets and φ_t a flow on X. The set A is called *chain-recurrent* with respect to B if for every $y_0 \in A$ and every $\varepsilon > 0$, $T > 0$ there exists a positive number $n \in \mathbb{N}$, a sequence of times $t_0, \ldots, t_{n-1} \geq T$, and points $y_1, \ldots, y_{n-1} \in B$ such that

$$\operatorname{dist}(\varphi_{t_i}(y_i), y_{i+1}) < \varepsilon, \qquad i = 0, \ldots, n-1 \quad (\text{mod } n, \text{ i.e. } y_n := y_0).$$

The set A is called *recurrent*, if we can chose chains of length $n = 1$ for all points, i.e. if $y_0 \in \omega(y_0)$ for all $y_0 \in A$.

Prove: For any $x_0 \in X$, the ω-limit $\omega(x_0)$ is chain-recurrent with respect to X, but it is not necessarily recurrent.

Free extra: Let the trajectory $\varphi_t(x_0)$ be bounded. Prove or disprove: The ω-limit $\omega(x_0)$ is chain-recurrent with respect to *itself*?

Exercise 38: Consider the autonomous differential equation

$$\dot{x} = f(x), \quad x \in \mathbb{R}^N$$

with Lipschitz-continuous f. Suppose that all $x \in \mathbb{R}^N$ satisfy the inequality

$$(*) \qquad f(x)^T x \geq \|x\|^3_{\mathbb{R}^N}.$$

Prove: The maximal existence time $t_+(x_0)$ of the solution $x(t)$ is bounded, for every non-zero initial condition $x_0 \neq 0$.

Free extra: Is $x = 0$ automatically an equilibrium if $(*)$ is satisfied?

Exercise 39: Choose at least one variant:

(A) In a bounded planar classroom, a professor chases a smart student. Both run at constant velocity 1. Can the professor reach the student?

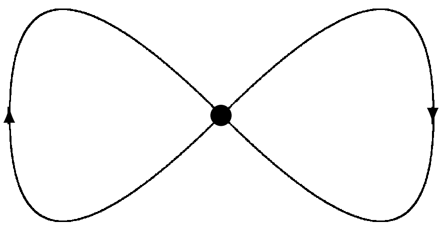
(B) The student runs along a circle, with velocity 1. The professor starts at the center and keeps running towards the momentary position of the student, at velocity $p < 1$. What is the professor's ω-limit set? Is the ω-limit set stable? What happens for $p = 1$, $p > 1$?

Hint (B): Useful coordinates are the angle between origin and professor, as seen from the student, and the distance between student and professor.

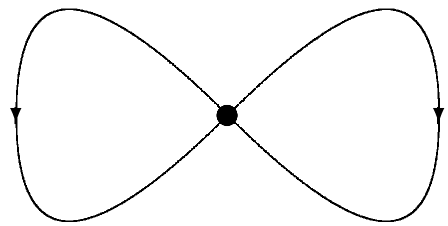
Free extra: Chase the professor.

Exercise 40: Which of the following sets are possible ω-limits of a trajectory of some planar flow? Which of the sets cannot occur as ω-limits? Justify your claims, without providing explicit vector fields.

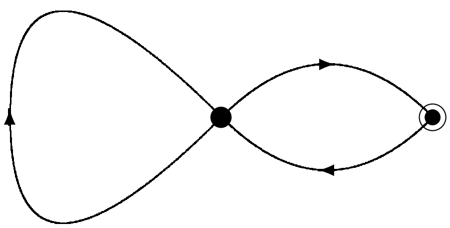
(i)



(ii)



(iii)



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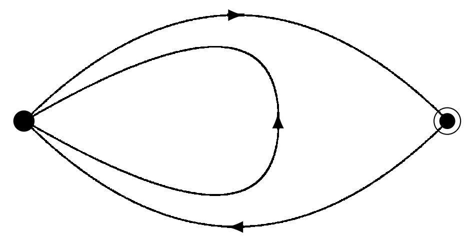

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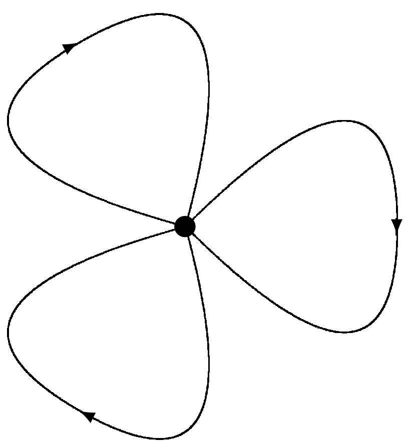
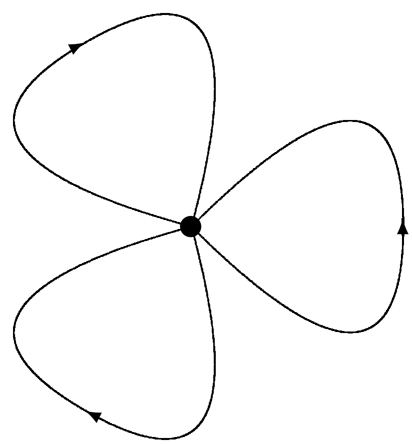


(vi)



Simple discs ● denote equilibria (of any type) and discs with a circle ⦿ denote hyperbolic saddles.