Completely Voluntary Holiday Problems
(for those who are not yet busy enough)

AM 0219: Nonlinear Dynamical Systems

Bernold Fiedler, Stefan Liebscher

Problem X1: Use your favorite numerical integrator (e.g. `dstool`) to plot the "time-2π map" (Poincaré map)

$$\begin{array}{rcl} P : S^1 \times \mathbb{R} & \to & S^1 \times \mathbb{R} \\ \begin{pmatrix} x(0) \\ \dot{x}(0) \end{pmatrix} & \mapsto & \begin{pmatrix} x(2\pi) \\ \dot{x}(2\pi) \end{pmatrix} \end{array}$$

of the swing ($x \approx 0$), respectively the inverted pendulum ($x \approx \pi$)

$$\ddot{x} + (\omega^2 + \alpha^2 \sin t) \sin x \;=\; 0.$$

a) near $x = 0$, $\dot{x} = 0$ width $\alpha = 0.2$ and $\omega = 0.05$ resp. $\omega = 0.5$.

b) near $x = \pi$, $\dot{x} = 0$ width $\alpha = 0.2$ and $\omega = 0.05$ resp. $\omega = 0.005$. (Here you should reduce the displayed region to $|\dot{x}| \leq 0.03$!).

Examine the stability properties of the swing / pendulum.

Hint if using ***dstool****:* At the Division of Applied Mathematics, connect via `ssh` (Secure Shell) to host `haiyang` and execute `dstool`.

One possibility to set up the equations in the parser (menu `Models` → `Parser...`) is:

```
♯ swing
x' = dx
dx' = -sin(x)*(w*w+a*a*sin(t))
t' = 1
INITIAL x 0.0 dx 0.3 t 0 w 0.05 a 0.2
RANGE x 0 6.283 t 0 6.283 dx -1 1
PERIODIC x 0 6.283 t 0 6.283
```

Use **Stopping condition** → **Poincare section** $t = 0$ in submenu **Panels** → **Orbits...**. You should decrease the number of periods (parameter **Stop**, standard value 5000) to 100. Otherwise you will need a lot of time. Then you can open **Panels** → **Two-D...**, chose initial contitions with the left mouse button, and start the iteration with the middle one. Use **Panels** → **Selected...** to change parameters.

Problem X2: [Arnol'd] We want to prove analytically that the upper (unstable) equilibrium of the pendulum can be stabilized by vertical vibrations.

Let l be the length of the pendulum. The vertical forcing has amplitude $a \ll l$ and period 2τ. As a simplification, we assume that the acceleration is piecewise constant, $\pm c = \pm 8a/\tau^2$.

The resulting equation of motion yields

$$\ddot{x}(t) = (\omega^2 \pm \alpha^2) x(t),$$

with $\omega^2 = g/l$ and $\alpha^2 = c/l$. The sign switches with period 2τ. Thus $\alpha^2 = 8a/(l\tau^2) > \omega^2$, for a sufficiently fast forcing ($\tau \ll 1$).

What is the minimal forcing frequency to stabilize the inverted pendulum?

Problem X3: Prove or disprove for vector fields f in $\mathbb{R}^3$: If $\operatorname{div} f \equiv 0$ then there exists a non-trivial first integral of f.

Here, a first integral I is called non-trivial if ∇I vanishes only at zeros of the vector field.

Problem X4: [Arnol'd, (Russian) sample examination problems] To stop a boat at a dock, a rope is thrown from the boat which is then wound around a post attached to the dock. What is the breaking force on the boat if the rope makes 3 turns around the post, if the coefficient of friction of the rope around the post is 1/3, and if a dockworker pulls at the free end of the rope with a force of 100 N (approximately the force one needs to lift $22\frac{1}{2}$ pounds of potatoes)?

Problem X5: [Arnol'd, (Russian) sample examination problems] It is known from experience that when light is refracted a the interface between media, the sines of the angles formed by the incident and refracted rays with the normal to the interface are inversely proportional to the indices of refraction of the media:

$$\frac{\sin \alpha_1}{\sin \alpha_2} = \frac{n_2}{n_1}.$$

Find the form of the light rays in the plane $(x, y) \in \mathbb{R}^2$ if the index of refraction is $n = n(y)$. Study the case $n(y) = 1/y$.

Remark: The half plane $\{y > 0\}$ with the index of refraction $n(y) = 1/y$ gives a model of Lobachevskian geometry.

Problem X6: [Arnol'd, (Russian) sample examination problems] Draw the rays emanating in different directions from the origin in a plane with index of refraction $n = n(y) = y^4 - y^2 + 1$.

Remark: This explains the formation of a mirage: the air over a desert has its maximum of refraction in a certain finite height, due to more rarefied air in higher and lower layers.

Acoustic channels in the ocean are a similar phenomenon: the maximum of rarefaction is found at a depth of 500m-1000m.

Problem X7: Can matrices $A, B \in \mathbb{R}^{n\times n}$ fail to commute if $e^A = e^B = e^{A+B} = \mathrm{id}$?

Problem X8: Can an unstable equilibrium position become stable upon linearization? Can it become asymptotically stable? Can an asymptotically stable equilibrium become unstable?

Problem X9: Let $y(t)$ be a continuous function solving the integral equation

$$y(t) = \int_0^t y(s)^\alpha \sin(2005\, y(s))\, \mathrm{d}s,$$

for some fixed $\alpha \geq 0$ and all $t \in [0, 1]$.

Prove: $y(t) \equiv 0$ is constant for all $t \in [0, 1]$. For which $\alpha \in [-1, 0)$ does the claim hold ?

Joyful Holidays,

a manageable amount of snow,

and a happy & peaceful New Year!