Exercises

AM 0220: Nonlinear Dynamical Systems

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Problem 1: Let $0 < \alpha < 1$ be irrational and

$$s_n \;:=\; \mathrm{sign}(\sin(n\pi\alpha)), \qquad n = 1, 2, 3, ...$$

The sequence

$$w_n \;:=\; |s_n - s_{n+1}|/2$$

detects the sign changes of the sequence s_n. Prove:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} w_n \;=\; \alpha.$$

Free extra: Is it possible to recover a rational number α from the sequence s_n?

Problem 2: Consider the vector field

$$\dot{y} = f(y), \qquad y \in S^1, \quad f(y) > 0$$

with corresponding flow φ_t. Determine a formula for the rotation number $\varrho(\varphi_{2\pi})$ of the time-2π-map, for example by separation of variables.

Problem 3: Calculate the rotation number $\varrho(\alpha)$ of the time-2π-map of the differential equation

$$\dot{x} = \alpha + \sin(x - t), \qquad x \in S^1.$$

Problem 4: Let $f : S^1 \to S^1$ be a homeomorphism of the circle that reverses orientation.

Prove: the rotation number $\varrho(f)$ vanishes. (Define the rotation number just as for orientation preserving homeomorphisms.)