Exercises

AM 0220: Nonlinear Dynamical Systems

Bernold Fiedler, Stefan Liebscher

due date: Mon, Feb 28, 2005

Problem 5: As stated in class, each flow φ_t on the 2-torus without equilibria and periodic orbits possesses a global transverse section S, i.e. a closed curve that intersects every trajectory and is everywhere transverse to the vector field.

Find an example of a flow φ_t on the 2-torus with (two or more) periodic orbits — but without equilibria — such that no global transverse section S exists.

Problem 6: Consider the map $A : \mathbb{R} \to \mathbb{R}$,

$$A(y) = \begin{cases} 2y, & 0 \leq y < 1 \\ A(y-1)+1, & 1 \leq y \\ A(y+1)-1, & y < 0 \end{cases}$$

Thus $A(y+1) = A(y)+1$ for all y and A defines a map $\tilde{A} : S^1 \to S^1 = \mathbb{R}/\mathbb{Z}$. However A and $\tilde{A}$ are not homeomorphisms. Nonetheless, try to define a "rotation number" $\varrho(y_0)$ for initial conditions y_0. Does $\varrho(y_0)$ depend on y_0?

Problem 7: Find an example to show that a homeomorphism f of the circle with rational rotation number $\varrho(f)$ is not necessarily conjugated to a rigid rotation by the angle $\varrho(f)$.

Give an additional — necessary and sufficient — condition on f to be conjugated to a rigid rational rotation.

Problem 8: Consider a diffeomorphism $\Phi : \mathbb{R}^N \to \mathbb{R}^N$ with $\Phi(0) = 0$. Let

$$\begin{aligned} W^s &= \{x \in \mathbb{R}^N \mid \lim_{n\to\infty} \Phi^n(x) = 0\} \\ W^u &= \{x \in \mathbb{R}^N \mid \lim_{n\to\infty} \Phi^{-n}(x) = 0\} \end{aligned}$$

denote the stable and the unstable set of the origin. Find — if possible — an example and a counterexample for each of the following cases:

(i) W^s is an embedded submanifold.

(ii) W^s is closed.

(iii) $W^s \cap W^u$ consists of exactly two distinct points.

Free extra: Find an example such that W^s is not even a (topological) manifold.