Exercises

# AM 0220: Nonlinear Dynamical Systems

Bernold Fiedler, Stefan Liebscher

**due date: Mon, Mar 07, 2005**

**Problem 9:** Calculate explicitly the stable and unstable manifolds of the pendulum

$$\ddot{\varphi} + \sin\varphi \;=\; 0$$

at the equilibrium $\varphi = \pi$, $\dot{\varphi} = 0$ for $\varphi \in \mathbb{R}$ and for $\varphi \in S^1$.

**Problem 10:** Consider the damped pendulum

$$\ddot{\varphi} + \alpha\dot{\varphi} + \sin\varphi \;=\; 0,$$

with $\alpha > 0$ and $\varphi \in \mathbb{R}$, see also problem 9.

(i) Sketch the stable manifold of the equilibrium $\varphi = \pi$, $\dot{\varphi} = 0$ for $\alpha > 0$.

(ii) How do trajectories above and below the stable manifold differ?

**Problem 11:** Calculate all fixed points of the bouncing-ball map $f$:

$$\begin{aligned} \Phi_{j+1} &= \Phi_j + v_j, \\ v_{j+1} &= \alpha\, v_j - \gamma\cos(\Phi_j + v_j) \end{aligned}$$

with $\Phi_j \in S^1 = \mathbb{R}/(2\pi\mathbb{Z})$ and $v_j \in \mathbb{R}$, for $0 < \alpha < 1$ and $0 < \gamma$. How many fixed points does $f$ have for given $\alpha, \gamma$? Determine the type (stable, unstable, non-hyperbolic etc.) of the fixed points. Sketch the dependence of the fixed points on $\gamma$, for $\alpha = \frac{1}{2}$. What happens for $\alpha \to 1$?

**Problem 12:** Let

$$\Sigma_N \;=\; \left\{ s = (s_j)_{j\in\mathbb{Z}} \mid s_j \in \{0, \ldots, N-1\} \right\}$$

be the set of sequences on $N$ symbols, with the metric

$$\operatorname{dist}(s, s') \;:=\; \sum_{j\in\mathbb{Z}} (2N)^{-|j|} |s_j - s'_j|.$$

Consider the shift

$$\sigma \;:\; \Sigma_N \to \Sigma_N, \quad (s_j)_{j\in\mathbb{Z}} \mapsto (s_{j+1})_{j\in\mathbb{Z}}.$$

What are the fixed points of $\sigma$ ? Determine the stable and unstable sets and thus all homoclinic and heteroclinic orbits of these fixed points.