Exercises

AM 0220: Nonlinear Dynamical Systems

Bernold Fiedler, Stefan Liebscher

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Problem 9: Calculate explicitly the stable and unstable manifolds of the pendulum

$$\ddot{\varphi} + \sin\varphi \;=\; 0$$

at the equilibrium $\varphi = \pi$, $\dot{\varphi} = 0$ for $\varphi \in \mathbb{R}$ and for $\varphi \in S^1$.

Problem 10: Consider the damped pendulum

$$\ddot{\varphi} + \alpha\dot{\varphi} + \sin\varphi \;=\; 0,$$

with $\alpha > 0$ and $\varphi \in \mathbb{R}$, see also problem 9.

(i) Sketch the stable manifold of the equilibrium $\varphi = \pi$, $\dot{\varphi} = 0$ for $\alpha > 0$.

(ii) How do trajectories above and below the stable manifold differ?

Problem 11: Calculate all fixed points of the bouncing-ball map f:

$$\begin{aligned} \Phi_{j+1} &= \Phi_j + v_j, \\ v_{j+1} &= \alpha\, v_j - \gamma\cos(\Phi_j + v_j) \end{aligned}$$

with $\Phi_j \in S^1 = \mathbb{R}/(2\pi\mathbb{Z})$ and $v_j \in \mathbb{R}$, for $0 < \alpha < 1$ and $0 < \gamma$. How many fixed points does f have for given α, γ? Determine the type (stable, unstable, non-hyperbolic etc.) of the fixed points. Sketch the dependence of the fixed points on γ, for $\alpha = \frac{1}{2}$. What happens for $\alpha \to 1$?

Problem 12: Let

$$\Sigma_N \;=\; \left\{ s = (s_j)_{j\in\mathbb{Z}} \mid s_j \in \{0, \ldots, N-1\} \right\}$$

be the set of sequences on N symbols, with the metric

$$\mathrm{dist}\,(s, s') \;:=\; \sum_{j\in\mathbb{Z}} (2N)^{-|j|} |s_j - s'_j|.$$

Consider the shift

$$\sigma \;:\; \Sigma_N \to \Sigma_N, \quad (s_j)_{j\in\mathbb{Z}} \mapsto (s_{j+1})_{j\in\mathbb{Z}}.$$

What are the fixed points of σ ? Determine the stable and unstable sets and thus all homoclinic and heteroclinic orbits of these fixed points.