Exercises

AM 0220: Nonlinear Dynamical Systems

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due date: Mon, Mar 14, 2005

Problem 13: Let $\Phi : M \to M$ be a continuous map on a metric space M. We call a sequence $(\xi_k)_{k\in\mathbb{N}}$ a δ-*pseudo orbit*, if the estimate

$$\mathrm{dist}(\Phi(\xi_k), \xi_{k+1}) \; < \; \delta.$$

holds for all $k \in \mathbb{N}$. We call a Φ-orbit $(x_k)_{k\in\mathbb{N}} = (\Phi^k(x_0))_{k\in\mathbb{N}}$ in M an ε-*shadow* of the pseudo orbit $(\xi_k)_{k\in\mathbb{N}}$ if the estimate

$$\mathrm{dist}(x_k, \xi_k) \; < \; \varepsilon$$

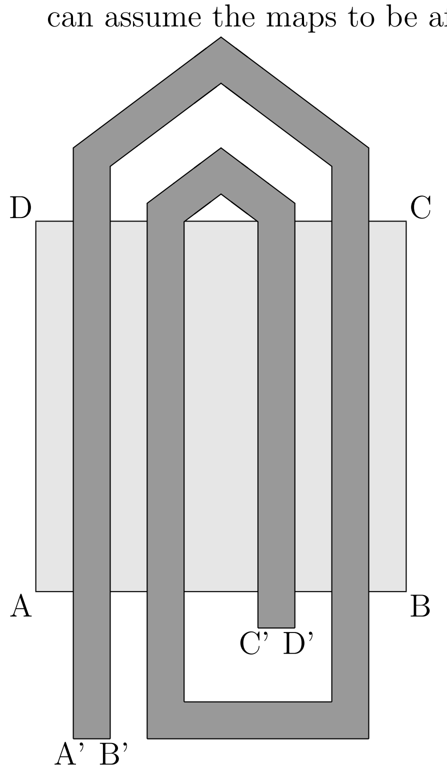
holds for all $k \in \mathbb{N}$. We say that the pair (M, Φ) has the *shadow property*, if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that every δ-pseudo orbit has an ε-shadow.

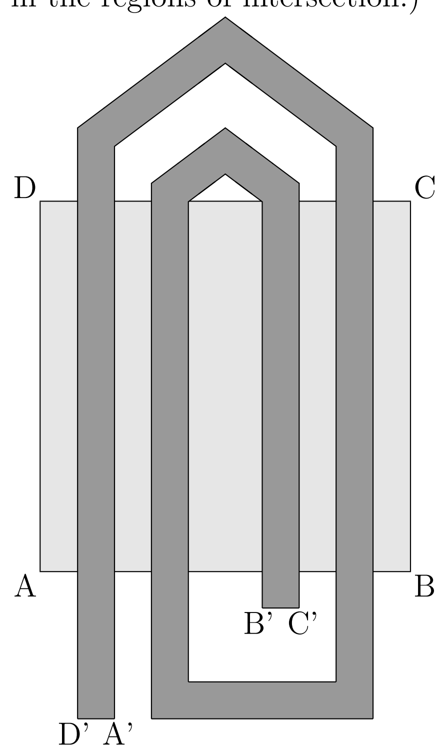
(i) Prove: the shift on two symbols has the shadow property.

(ii) Give an interpretation of the shadow property from a numerical view point.

Free extra: Is the shadow unique?

Problem 14: Which of the following "paper-clip" maps gives rise to shift dynamics? (You can assume the maps to be affine linear, in the regions of intersection.)





Problem 15: Consider the map

$$f\,:\,S^1 \to S^1 \;=\; \mathbb{R}/\mathbb{Z}, \qquad f(y) \;:=\; 2y \pmod 1,$$

see also Problem 6. Prove that f contains periodic orbits of every period. Are there orbits which are dense on the circle S^1?

Free extra: What can you say about $f(y) = gy \pmod 1$ for integer factors $g \geq 2$?

Problem 16: Consider again the bouncing-ball map $f_{\alpha,\gamma}$ on $S^1 \times \mathbb{R}$:

$$\begin{aligned} \Phi_{j+1} &= \Phi_j + v_j, \\ v_{j+1} &= \alpha\, v_j - \gamma \cos(\Phi_j + v_j), \end{aligned}$$

with $0 < \alpha < 1$ and $0 < \gamma$, and the domain

$$D \;:=\; \left\{ (\Phi, v) \in S^1 \times \mathbb{R} \;:\; |v| \leq \frac{\gamma}{1-\alpha} + \varepsilon \right\}$$

for some $\varepsilon > 0$. As in class, define the attractor

$$\mathcal{A}_{\alpha,\gamma} \;:=\; \bigcap_{n=0}^{\infty} f^n_{\alpha,\gamma}(D).$$

(i) Prove that $\mathcal{A}_{\alpha,\gamma}$ does not depend on the choice of ε.

(ii) Is $\mathcal{A}_{\alpha,\gamma}$ connected?

(iii) Prove that the map $(\alpha, \gamma) \mapsto \mathcal{A}_{\alpha,\gamma}$ is upper-semicontinuous, i.e.

$$\lim_{(\alpha,\gamma)\to(\bar\alpha,\bar\gamma)} \operatorname{dist}\left(\mathcal{A}_{\alpha,\gamma}, \mathcal{A}_{\bar\alpha,\bar\gamma}\right) \;=\; 0,$$

for the (non-symmetric) distance

$$\operatorname{dist}(A, B) \;=\; \sup_{x\in A}\, \inf_{y \in B} |x - y|.$$

What is the interpretation of upper-semicontinuity in this case?