Exercises

AM 0220: Nonlinear Dynamical Systems

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Problem 17: Let the assumptions of theorem III.2.2 (C^0-horseshoe) be satisfied for the iteration Φ on the square Q. Thus, there exists a homeomorphism τ conjugating the shift $\sigma : S \to S$ to $\Phi : I \to I$, on an invariant subset $I := \tau(S) \subset Q$. Let the horizontal and vertical Lipschitz-curves $U(s)$ and $V(s)$ be defined as in class, that is

$$\begin{aligned} U(s) &:= \left\{ q \in Q \;\middle|\; \Phi^{-k}(q) \in V_{s_k} \;\; \forall k \geq 1 \right\}, \\ V(s) &:= \left\{ q \in Q \;\middle|\; \Phi^{-k}(q) \in V_{s_k} \;\; \forall k \leq 0 \right\}, \end{aligned}$$

for any sequence $s = (s_k)_{k \in \mathbb{Z}} \in S$ and the primary vertical stripes $\{V_a \,|\, a \in A\}$ of the horseshoe construction.

Consider the unstable and stable sets of points $p \in I$,

$$\begin{aligned} W^{\mathrm{u}}(p) &:= \left\{ q \in Q \;\middle|\; \Phi^k(q) \in \bigcup_{a \in A} V_a \;\; \forall k \leq -1, \;\; \lim_{k \to -\infty} \operatorname{dist}\left(\Phi^k(p), \Phi^k(q)\right) = 0 \right\} \\ W^{\mathrm{s}}(p) &:= \left\{ q \in Q \;\middle|\; \Phi^k(q) \in \bigcup_{a \in A} V_a \;\; \forall k \geq 0, \;\; \lim_{k \to \infty} \operatorname{dist}\left(\Phi^k(p), \Phi^k(q)\right) = 0 \right\}. \end{aligned}$$

Let $p = \tau(s)$ be a point of the invariant set with corresponding sequence s. Prove:

(i) $U(s) \subset W^{\mathrm{u}}(p)$ and $V(s) \subset W^{\mathrm{s}}(p)$;

(ii) $U(s)$ and $V(s)$ are the connected components of $W^{\mathrm{u}}(p)$ and $W^{\mathrm{s}}(p)$ containing p. Thus, they are the local unstable and stable (Lipschitz) manifolds of p.

Problem 18: Let the assumptions of theorem III.2.4 be satisfied for the C^1-iteration Φ on the square Q. Thus, there exists a horseshoe $I \subset Q$ with hyperbolic structure.

Prove that all periodic points of Φ on I are hyperbolic saddles of the period-map.

Problem 19: Let φ_t be a flow and $\Phi = \varphi_1$ the corresponding time-1 map. Prove that there are no transverse homoclinic points of Φ.

Problem 20: [(infinite) adding machine] Consider the space

$$\Sigma_2^+ = \left\{ s = (s_j)_{j \in \mathbb{N} \cup \{0\}} \;\middle|\; s_j \in \{0,1\} \right\}$$

of (one sided) sequences on the two symbols $\{0,1\}$. The topology on Σ_2^+ is the product topology, as for the space of two-sided sequences defined in class. It is generated by the cylinder sets

$$N_k(s) := \left\{ \tilde{s} \in \Sigma_2^+ \;\middle|\; \tilde{s}_j = s_j \;\; \forall j \le k \right\}, \qquad s \in \Sigma_2^+, \quad k \in \mathbb{N},$$

and an equivalent metric would be

$$\operatorname{dist}(s, \tilde{s}) := \sum_{j \ge 0} 2^{-j} \left| s_j - \tilde{s}_j \right| .$$

For an arbitrary but fixed element $g \in \Sigma_2^+$ define the (addition) map

$$T_g : \Sigma_2^+ \longrightarrow \Sigma_2^+, \qquad a = (a_n)_{n \in \mathbb{N} \cup \{0\}} \longmapsto T_g(a) = (T_g(a)_n)_{n \in \mathbb{N} \cup \{0\}}$$

recursively by

$$\begin{aligned} r_0 &:= 0, \\ T_g(a)_n + 2r_{n+1} &:= a_n + g_n + r_n, \qquad n = 0, 1, \ldots \end{aligned}$$

Here the temporary variable $r \in \Sigma_2^+$ denotes the overflow of the addition. If one takes *finite* sequences as binary representations of integer numbers then $T_g(a)$ is just the sum of a and g.

(i) Let $g_0 = 0$. Prove that Σ_2^+ does not contain a dense orbit of T_g. (Find, for example, nontrivial invariant subsets of Σ_2^+.)

(ii) Let $g = (1, 0, 0, 0, \ldots)$. Find a dense orbit.

Free extra: Prove that T_g has a dense orbit for arbitrary g with $g_0 = 1$. In this case, *each* orbit of T_g is dense in Σ_2^+. However there are no periodic orbits and no sensitive dependence on initial conditions, thus no “chaos”.