Exercises

# AM 0220: Nonlinear Dynamical Systems

Bernold Fiedler, Stefan Liebscher

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**Problem 21:** [Proof of hyperbolic horseshoe] Consider a $C^1$-iteration $\Phi$ on the square $Q$ satisfying the assumptions of the theorem on the $C^1$-horseshoe. In particular, this includes a positive parameter $\mu < 1/2$ and forward/backward invariant cones $S^+ = \{(\xi, \eta) \,:\, |\eta| \le \mu|\xi|\}$, $S^- = \{(\xi, \eta) \,:\, |\xi| \le \mu|\eta|\}$,

$$D\Phi(p)\, S^+ \subset \operatorname{int} S^+ \cup \{0\}, \qquad D\Phi^{-1}(p)\, S^- \subset \operatorname{int} S^- \cup \{0\},$$

for each $p \in \bigcup_{a\in A} V_a$, with the expansion/contraction properties

$$\begin{aligned} |\xi_1| &\ge \mu^{-1}|\xi| \qquad \text{for all} \quad (\xi, \eta) \in S^+, \\ |\eta_1| &\le \mu|\eta| \qquad \text{for all} \quad (\xi_1, \eta_1) \in S^-, \end{aligned}$$

with $\binom{\xi_1}{\eta_1} = D\Phi(p) \binom{\xi}{\eta}$. Thus there exists a horseshoe on some invariant set $I \subset Q$. Assume additionally

$$\mu^2 < \inf_{p\in I} |\det D\Phi(p)| \quad \text{and} \quad \mu^2 < \inf_{p\in I} |\det D\Phi^{-1}(p)|.$$

Prove: there exists a unique hyperbolic structure on $I$.

*Hint:* Consider line bundles $L^{\pm}(p)$ in $S^{\pm}$ on $I$, say given by $\eta = \alpha^+(p)\xi$ and $\xi = \alpha^-(p)\eta$ with $|\alpha^{\pm}| \le \mu$. Then prove that the action of $D\Phi^{\pm 1}$ on $L^{\pm}$ is a contraction.

**Problem 22:** Consider the iteration on the 2-torus $T = (\mathbb{R}/\mathbb{Z})^2$ defined by the matrix

$$B = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

Find a horseshoe for a suitable iterate $B^k$, $k > 0$.

*Hint:* Identify the torus with the unit square centered at $(0, 0)$ and investigate the images of a parallelogram parallel to the eigenvectors of $B$.

**Problem 23:** We want to "rescue" the Poincaré-Bendixson theorem for differential equations in $\mathbb{R}^3$. Thus we allow three possibilities for the $\omega$-limit of a point $x$:

(i) an invariant torus,

(ii) a periodic orbit,

(iii) $\alpha(y)$ and $\omega(y)$ contain only equilibria for each $y \in \omega(x)$.

Why is this variant still wrong? (Find a counterexample.)

**Problem 24:** Consider the Hénon map

$$\begin{aligned} x_{j+1} &= 1 - \alpha x_j^2 + \beta y_j, \\ y_{j+1} &= x_j. \end{aligned}$$

Find a horseshoe for $1 \ll \alpha$ and $0 < \beta \ll 1$.

*Hint:* $Q = [-0.1, 0.1] \times [-1, 1]$.