Exercises

AM 0220: Nonlinear Dynamical Systems

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Problem 25: Show that the system

$$\begin{aligned}\dot{x} &= x^2,\\ \dot{y} &= -y,\end{aligned}$$

has infinitely many local center manifolds of $(x, y) = (0, 0) \in \mathbb{R}^2$.

Free extra: Modify the vector field outside a neighborhood of the origin to obtain any of these local center manifolds as a global one.

Problem 26: Consider the system of differential equations

$$\begin{aligned}\dot{x} &= xy,\\ \dot{y} &= -y - x^2.\end{aligned}$$

Use a (local) center manifold to decide whether the equilibrium $x = y = 0$ is asymptotically stable.

Note: Use the invariance of the center manifold to calculate the necessary terms of its Taylor expansion.

Problem 27: Let Φ be a diffeomorphism of the plane $\mathbb{R}^2$ with a transverse homoclinic orbit. In class, we found a shift on two symbols for an iterate Φ^n. Prove that for every $m \in \mathbb{N}$ the sift of m symbols is conjugate to some iterate Φ^n on a suitable subset of $\mathbb{R}^2$.

Problem 28: A flow φ_t on $\mathbb{R}^N$ is called *dissipative*, if there exists a bounded set $B \subset \mathbb{R}^N$ which absorbs every trajectory, i.e.

$$\forall\, x \in \mathbb{R}^N \;\; \exists\, t_0(x) \in \mathbb{R} \;\; \forall\, t > t_0(x) \quad \varphi_t(x) \in B.$$

Consider a dissipative flow φ_t. Prove that the set

$$A := \{\, x \in \mathbb{R}^N \mid \text{the orbit } \{\varphi_t(x) \,|\, t \in \mathbb{R}\} \text{ is bounded} \,\}$$

(i) is non-empty;

(ii) is invariant under φ_t;

(iii) is itself bounded;

(iv) is attracting, i.e. $\lim_{t\to\infty} \operatorname{dist}(\varphi_t(x), A) = 0$ for all $x \in \mathbb{R}^N$.

Free extra: The set A is an attractor, it is *the global attractor of* φ_t.