Exercises

AM 0220: Nonlinear Dynamical Systems

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Problem 29: Consider the parameter-dependent vector field

$$\begin{aligned} \dot{x} &= 2(x+y)^2 + x - y - \lambda, \\ \dot{y} &= \qquad\qquad -x + y - \lambda. \end{aligned}$$

Calculate the local center manifold of the origin and the reduced vector field (that is their Taylor expansions of sufficient order) to determine the bifurcation diagram.

Problem 30: Consider a vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ of the flow φ_t with equilibrium $x_0 = 0$. Let $R : \mathbb{R}^n \to \mathbb{R}^n$ be a linear involution ($R \circ R = \mathrm{id}$). Assume that the vector field f is equivariant under the symmetry R, that is

$$f \circ R \;=\; R \circ f.$$

(i) Let the assumptions of the theorem on the existence of a *global* center manifold in $x_0 = 0$ be satisfied. Prove that the center manifold $W^c(x_0)$ inherits the symmetry of the vector field, that is
$$R(W^c) \;=\; W^c.$$

(ii) Let the assumptions of the theorem on the existence of a *local* center manifold in $x_0 = 0$ be satisfied. Prove that there exists a symmetric local center manifold $W^c_{\mathrm{loc}}(x_0)$, i.e.
$$R(W^c_{\mathrm{loc}}) \;=\; W^c_{\mathrm{loc}}.$$

Free extra: The reduced vector field on the (symmetric) center manifold inherits the symmetry, i.e. it is equivariant under R restricted to the center eigenspace.

Problem 31: Consider a parameter-dependent vector field

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} \;=\; f(x,y;\lambda) \;=\; \begin{pmatrix} \lambda & 0 \\ 0 & B(\lambda) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + g(x,y;\lambda),$$

with $x \in \mathbb{R}$, $y \in \mathbb{R}^{N-1}$, some hyperbolic matrix $B(\lambda)$, and terms $g(x,y;\lambda) = \mathcal{O}(|x|^2 + \|y\|^2)$ of higher order. Assume that f is symmetric under the reflection at the y-plane,

$$f \circ R \;=\; R \circ f \qquad \text{with} \qquad R \;=\; \mathrm{diag}(-1, 1, \ldots, 1).$$

Discuss the reduced vector field on a local center manifold.

Problem 32: A measure of complexity of a map Φ is the *topological entropy* h: Let $N(n)$ be the number of periodic points of Φ with (not necessarily minimal) period n. Then the entropy is defined as

$$h := \limsup_{n\to\infty} \frac{\log N(n)}{n}.$$

Calculate the entropy h of the shift on m symbols. Prove that every iteration Φ containing a shift (i.e. there exists an invariant set I such that $\Phi|_I$ is conjugate to a shift on m symbols) has positive topological entropy.