Exercises

AM 0220: Nonlinear Dynamical Systems

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Problem 33: Consider the system

$$\begin{aligned} \dot{x}_{\mathrm{c}} &= Ax_{\mathrm{c}} + f(x_{\mathrm{c}} + x_{\mathrm{h}}), \\ \dot{x}_{\mathrm{h}} &= Bx_{\mathrm{h}} + g(x_{\mathrm{c}} + x_{\mathrm{h}}), \end{aligned}$$

with $f, g \in C^\kappa$, $f(x) = \mathcal{O}(|x|^2)$, $g(x) = \mathcal{O}(|x|^2)$, and $\mathrm{spec}(A) \subset \mathbf{i}\mathbb{R}$, $\mathrm{spec}(B) \cap \mathbf{i}\mathbb{R} = \emptyset$. Assume the existence of a local center manifold $x_{\mathrm{h}} = h(x_{\mathrm{c}})$, $h \in C^\kappa$.

Prove that the κ-th derivative of h is uniquely determined. Describe a method to calculate the Taylor expansion of h.

Hint: Compare the Taylor expansions of $Bh(x_{\mathrm{c}}) + g(x_{\mathrm{c}} + h(x_{\mathrm{c}}))$ and $Dh(x_{\mathrm{c}})\,[Ax_{\mathrm{c}} + f(x_{\mathrm{c}} + h(x_{\mathrm{c}}))]$ and use the fact that $h(x_{\mathrm{c}}) = \mathcal{O}(|x_{\mathrm{c}}|^2)$.

Problem 34: Let $H_2(\mathbb{R}^2)$ be the space of homogeneous polynomials of degree 2 in the two variables x, y, that is

$$H_2(\mathbb{R}^2) \;=\; \mathrm{span}\left\{ \begin{pmatrix} x^2 \\ 0 \end{pmatrix}, \begin{pmatrix} xy \\ 0 \end{pmatrix}, \begin{pmatrix} y^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x^2 \end{pmatrix}, \begin{pmatrix} 0 \\ xy \end{pmatrix}, \begin{pmatrix} 0 \\ y^2 \end{pmatrix} \right\}.$$

Determine the image of $\mathrm{ad}\,A(H_2(\mathbb{R}^2))$ for the matrix

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Problem 35: On the space $M^{n\times n}(\mathbb{C})$ of $(n \times n)$-matrices, we define the scalar product $(A, B) := \mathrm{trace}\,(AB^*)$ and the commutator $L_A B = [A, B] = AB - BA$. Here $A, B \in M^{n\times n}(\mathbb{C})$ and B^* denotes the adjoint matrix $(B^*)_{ij} = \overline{(B)_{ji}}$.

(i) Prove: $(L_A)^* = L_{A^*}$, i.e. $(L_A B, C) = (B, L_{A^*} C)$ for all $B, C \in M^{n\times n}(\mathbb{C})$.

(ii) Calculate $\ker\,(L_A)^*$ for $A = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $A = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \mu \end{smallmatrix}\right)$, with $0 \neq \lambda \neq \mu \neq 0$.

(iii) Prove that the map $\Psi_A : GL(n) \to M^{n\times n}(\mathbb{C}),\ \ G \mapsto G^{-1}AG$ is analytic with derivative $D\Psi_A(\mathrm{id}) : M^{n\times n}(\mathbb{C}) \to M^{n\times n}(\mathbb{C}),\ \ C \mapsto L_A C$.

Problem 36: Discuss the “cusp bifurcation”

$$\dot{x} \;=\; x^3 + \lambda x + \mu, \qquad x, \lambda, \mu \in \mathbb{R}.$$

In particular, determine number and stability of equilibria for all parameters (λ, μ) and the parameter curves along which degenerate equilibria (i.e. saddle-node bifurcations) occur.