Self-similar measures

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Abstract. After an introduction to self-similar deterministic and random measures, we introduce an algorithm which decides on the separation condition for given similarity mappings defining a self-similar set. We construct some new types of Sierpinski gasket and determine their exponent of interior distance.

1 Introduction

Most of the well-known examples of fractals come from the early stage of set theory and general topology at about 1900. They were designed to clarify basic concepts like "continuum", "curve" and "one-dimensional space", by demonstrating that subsets of the line or plane can have very strange, counterintuitive properties. A Cantor set is uncountable yet contains no connected pieces. A simple curve can be nowhere differentiable, it can even have positive area. The Sierpinski gasket in figure 1 has topological dimension one. Nevertheless, each point is a branching point in the sense that it can be connected by three disjoint curves to the three vertices of the triangle.

The virtue of Mandelbrot's revival of these spaces with the label "fractals" was to consider them not as pathological counterexamples, but as models for real world phenomena. This idea was eagerly accepted by many people working in the experimental sciences. In particular the Sierpinski gasket S (figure 1) was used as a universal toy model for catalysts and other porous materials. A Laplacian on S was defined and its eigenvalues determined, first by physicists ([10] gives a review of part of the literature) and later with mathematical rigour, via Dirichlet forms as well as via an appropriate Brownian motion. This theory was generalized to a class of spaces termed "post-critically finite (p.c.f.) self-similar sets with harmonic structure" [17,21,12,13].

Unfortunately, there are very few concrete examples in this class. There is a real discrepancy between the well-developed "harmonic analysis on self-similar fractals" and the small number of concrete spaces for which this theory works. Here we try to improve the situation by presenting a method which constructs plenty of new p.c.f. spaces.

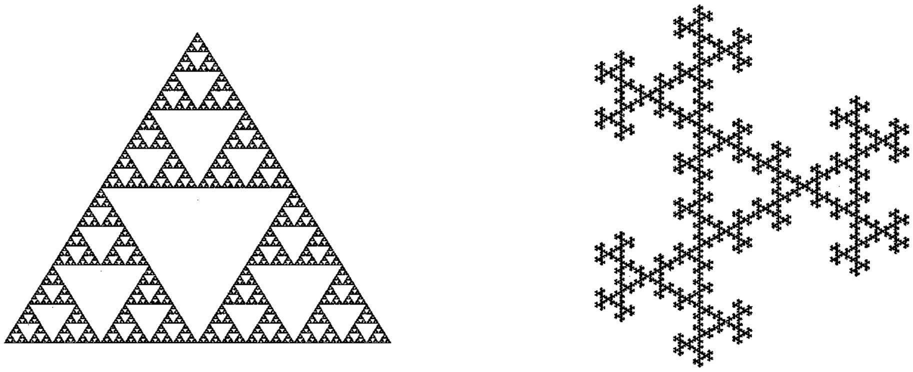

Fig. 1. The Sierpinski gasket S and its reversed version

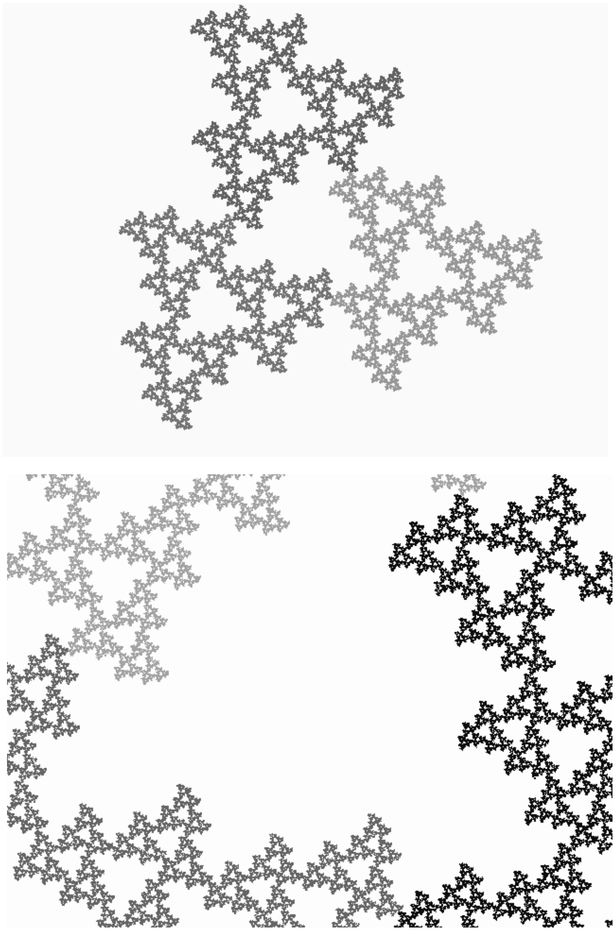

Fig. 2. A new gasket with a magnification

The set S, for instance, is still the standard example, but it has one disadvantage. Any two points of S can be connected by a curve of finite length, a kind of polygonal line. This should not be true in a proper fractal. In this paper we shall construct, by use of certain dynamical systems, the modification of S shown in figure 2. Its topological structure is very similar to that of S, but the shortest curves connecting different points are fractals themselves. They can be measured by Hausdorff measure of a certain dimension slightly larger than 1.

Before that we shall give a brief survey on self-similar deterministic and random measures and on some questions we have studied in the dynamics Schwerpunktprogramm.

2 Self-similar constructions

While no rigorous definition of the term "fractal" is generally accepted up to now, the intuitive idea of Mandelbrot was self-similarity: large and small pieces show the same appearance as the whole figure. A mathematical class of self-similar sets was introduced by Hutchinson in 1981, see [11,20]. A map $f : X \to X$ from a complete metric space (X, d) – usually Euclidean $\mathbb{R}^d$– into itself is called a similarity map with factor r if $d(f(x), f(y)) = r \cdot d(x, y)$ for all points x, y in X.

Given similarity maps $f_1, ..., f_m$ which are contractive, that is $r_i < 1$, a set $A \subseteq X$ is called *self-similar* with respect to the f_i if it solves the equation

$$A = f_1(A) \cup ... \cup f_m(A) .$$

The idea is that A consists of pieces which are geometrically similar to A. Substituting A on the right-hand side again by the union of pieces, we see that A consists of even smaller pieces $f_i f_j(A)$ etc. It is also rather easy to verify that among the compact subsets of X there is exactly one solution of the above equation.

Nevertheless, if one takes arbitrary similarity maps, the reulting A is not very spectacular. If the r_i are small, A will consists of tiny separate pieces – a very thin Cantor set. If the r_i are large then usually the pieces $f_i(A)$ will overlap so strongly that no pieces and no self-similarity can be recognized. Hutchinson studied a separation condition which guarantees that we can distinguish the pieces. It goes back to Moran who studied similar constructions without using mappings.

The *open set condition* says that there should exist an open set V such that the images $f_i(V)$ are pairwise disjoint and all contained in V.

One may imagine that V is like a neighbourhood of A and the $f_i(V)$ are like disjoint neighbourhoods of the pieces $f_i(A)$, which is true for Cantor sets. In general, V need not intersect A, and V is hard to find in cases like figure 3 on the left. For the Sierpinski gasket we have $f_i(x) = \frac{1}{2}(x + c_i)$ where the c_i are the three vertices of the triangle, and the interior of the triangle can be taken as V. For the other space in figure 1 we have $f_i(x) = \frac{1}{2}(-x + 3c_i)$ where the c_i denote the vertices of the big triangle in the middle, and again V can be taken as the interior of the convex hull of A, a certain hexagon. This is not true for figure 2, as can be seen in the magnification.

In figure 1 and 2, there are only finitely many points x in the intersections $A_i \cap A_j$. Moreover, when we look for preimages inside A of the form $x' = f_{i_1}^{-1}(x), x'' = f_{i_2}^{-1}(x')$ and so on, we get only finitely many points altogether. These points are considered as the boundary of the space – in figure 1 they are the vertices of V, for figure 2 they will be studied at the end of the paper. If the $A_i \cap A_j$ and the boundary are finite, the self-similar set is called post-critically finite, p.c.f. This term comes from papers on Julia sets.

The structure of self-similar sets is different from the attractors of usual dynamical systems, it rather resembles certain repellors. For example, the Julia set of the rational function $g(z) = \frac{2}{3} \cdot (z^2 - \frac{2}{z})$ is topologically identical to the Sierpinski gasket (which is not true for figure 2). On the other hand, the simplicity of the generating maps allows to study our sets in much greater detail as would be possible for hyperbolic repellors.

As for attractors and repellors, invariant measures play an important role. Given contractive similarity maps f_i and positive numbers p_i for $i = 1, ..., m$ with $p_1 + ...p_m = 1$, a Borel probability measure μ on X is called *self-similar measure* with respect to the f_i and p_i if

$$\mu = p_1 f_1(\mu) + ... + p_m f_m(\mu)$$

where $f_i(\mu)(B) = \mu(f_i^{-1}(B))$ denotes the image measure under f_i. Again there is an existence and uniqueness theorem for the solution. The choice of p_i allows to distribute the mass of μ unevenly over the pieces, which has become a standard toy model of so-called multifractals.

Since all these sets and measures look somewhat artificial, not like models of nature, there have been attempts to include randomness in the construction. The f_i and $p_i (i = 1, ..., m)$ can be chosen from a probability distribution Θ on $(Sim(X))^m \times S_m$ where $Sim(X)$ denotes the family of similarity maps and $S_m = \{(q_1, ..., q_m) | \, q_i > 0, \sum q_i = 1\}$. This should be done for each subpiece and piece of subpiece etc. in an independent and identically distributed manner. The result is a random measure – a distribution Ψ on the space $\mathcal{M}_1(X)$ of probability measures on X. Each μ is one random realization of Ψ. One example is shown in figure 3. There is again an equation describing self-similarity, not of single realizations, but of the whole random procedure:

$$\Psi = p_1 f_1(\Psi) + ... + p_m f_m(\Psi)$$

where $(f_1, ..., f_m, p_1, ..., p_m)$ is distributed with respect to Θ and $f_i(\Psi)$ means the image distribution on probability measures. This was worked out independently around 1986 by Falconer, by Graf and by Mauldin and Williams (cf. [11]). Recently, Hutchinson and Rüschendorf have simplified the existence and uniqueness theorem [16].

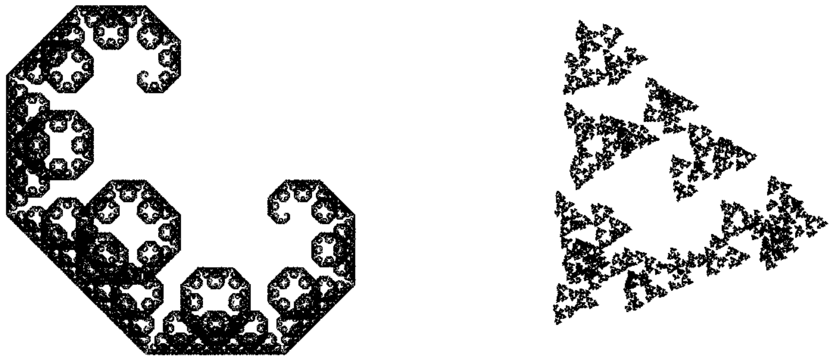

Fig. 3. Left: the Levy curve. Right: a random modification of S

However, there are several points which cast doubt on the usefulness of this constructive concept of random self-similarity. To get realizations with nice properties, the open set condition had to be assumed for almost all realizations, which seems a too stringent condition on the distribution Θ. So far, very few meaningful distributions Θ have been found. The randomness in the construction is unevenly distributed because the first choices decide on the overall appearance of the realization and later choices concern tiny details. Finally, one may wonder whether any fractals in nature (clouds, soil, Swiss cheese and other porous materials ...) are built by choice of random mappings.

3 Axiomatic approach to self-similarity

The author has suggested an axiomatic approach to random self-similarity [4,5] which uses ideas of Mandelbrot, U. Zähle and several other authors [25,9,19,24]. Let B denote the unit ball in $\mathbb{R}^d$. A random measure on B is a function $\tilde{\mu}$ from a certain probability space Ω into the set $\mathcal{M}_1(B)$ of probability measures on B. For our purpose, we do not need this function. We shall work with the distribution of the random variable $\tilde{\mu}$ which is a probability distribution on the set $\mathcal{M}_1(B)$ and will be called Ψ. We do not want to construct Ψ by similarity maps – we just want to say under which condition Ψ is self-similar.

Let us consider an arbitrary ball C inside B. *The condition will be that an observer who sees only the restrictions to C of all realizations μ of the distribution Ψ will see the same collection of pictures as somebody who observes all complete realizations μ on B.* This will be made precise.

Let h be the homothety mapping C onto B. If C has center c and radius ϵ, then $h(y) = (y - c)/\epsilon$. For a probability measure μ on B with $\mu(C) > 0$ let $h(\mu)$ denote the image measure under h of μ restricted to C. Then

$$H(\mu) = \frac{1}{\mu(C)} \cdot h(\mu)$$

is again a probability measure. Thus we have defined a mapping H from $\mathcal{M}_1(B)$ into itself which rescales the collection of pictures of the observer of C to the size of the ball B.

Let $\mathcal{C} = \{\mu \in \mathcal{M}_1(B) | \, \mu(C) > 0\}$, and let $H(\Psi)$ denote the image measure under H of Ψ restricted to $\mathcal{C}$. In other words, we disregard all μ which give an empty view for the observer of C. Now we define

$$\Psi_C = \frac{1}{\Psi(\mathcal{C})} \cdot H(\Psi) \ .$$

Definition 3.1. (Scale-invariance of random measures)
A random measure on the unit ball B in $\mathbb{R}^d$ with distribution Ψ is called scale-invariant if $\Psi_C = \Psi$ for each ball C inside B.

This setting may seem a bit unusual but we are convinced that it is the right way to look at the intuitive concept of self-similarity. We defined a self-similarity with respect to all similarity mappings with arbitrary factors. It does not involve any exponents. Nevertheless, we obtain the existence of at least two exponents.

Proposition 3.2. (The first exponent of a scale-invariant measure)
For every scale-invariant Ψ, there is a number α between 0 and d such that for every ball $C \subseteq B$ with radius r we have

$$\Psi(\mathcal{C}) = r^\alpha \ .$$

The proof is simple; it was given in [5] in terms of random sets. In typical examples, $d-\alpha$ is the Hausdorff dimension of the support *supp* μ for Ψ-almost all realizations μ. However, there are a few degenerate examples consisting of discrete measures for which this does not hold.

The second exponent of a scale-invariant measure needs some more preparation since it refers to the so-called Palm measure of Ψ. Roughly speaking, if B_r denotes the ball around zero with radius r and $h(r)$ denotes the "average value" of $\mu(B_r)$ then $h(r) = r^\beta$. The average is taken here with respect to Ψ but we have conditioning to those μ for which *all* B_r have positive measure, and we have to take geometric averages: $h(r) = \exp(\mathbf{E}(\log \mu(B_r)))$. Thus β is the parameter called pointwise dimension for ordinary measures.

Unfortunately, not too many examples of scale-invariant measures were found so far. We can take standardized occupation time measure of Brownian motion in $\mathbb{R}^d$ with $d \geq 3$, when we start the motion with a uniformly distributed point on the surface of B. We can also consider the zero set on the time axis of one-dimensional Brownian motion, or more generally stable subordinators, with a certain standardization. There are a few other probabilistic constructions [5] but there should be many more examples connected with spatial branching processes, continuous limits of interacting particle systems, with critical percolation clusters etc. This is difficult stuff, which has motivated us to direct our work more to the field of probability.

Recently, we found that at least the deterministic self-similar Hausdorff measures with open set V also fit into the picture of scale-invariance. The idea is to take the set A not as a complete figure, but rather as a basic space from which we can sample circular "views" of part of A [24]. One can consider "centered views" around points of A which lead to the so-called tangential distribution of self-similar measures [15] but here we study views inside V which are not centered. Details will be given in a forthcoming paper.

4 An algorithm deciding on separation

Now we are back to deterministic self-similar sets. The open set condition (OSC) was introduced already 1946 by Moran to prove that the Hausdorff measure $\mu^\alpha(A) > 0$ for the similarity dimension α of A. This is the unique number determined by $r_1^\alpha + ... + r_m^\alpha = 1$. Since clearly $\mu^\alpha(A) < \infty$, this means that μ^α is the natural volume function and the most simple self-similar measure on A.

Bandt and Graf [6] gave an algebraic equivalent condition. They considered the set of all maps

$$E = \{f_{i_n}^{-1} \cdot ... \cdot f_{i_1}^{-1} \cdot f_{j_1} \cdot ... \cdot f_{j_n} \mid i_1 \neq j_1, n \in N, 1 \leq i_k, j_k \leq m\}$$

Then $\mu^\alpha(A) > 0$ holds if and only if the identity map is not in the closure of E with respect to the pointwise convergence of maps on $\mathbb{R}^d$.

Schief [22] proved that this also equivalent to the OSC. Thus while originally the OSC seemed only a sufficient technical condition for positive Hausdorff measure, it has turned out that it is also necessary, and can be formulated in algebraic terms. It is the proper separation condition. While the open set is not easy to find even when you see the fractal, the algebraic condition $id \notin cl(E)$ can be used to decide on separation, given only the mappings f_i. Actually, this can be done by an algorithm indicated in [3] which can easily be implemented on a computer.

To describe the algorithm, we note that the elements of E can be generated in the form of a tree of mappings, with the identity map as root vertex. Consider the automorphisms

$$\Phi_{ij}(g) = f_i^{-1} \cdot g \cdot f_j \,, \qquad i,j = 1, ..., m$$

which act on the space of all similarity maps: $f \in Sim(\mathbb{R}^d)$. It is clear that the Φ_{ij} generate E, starting from $g = id$ and requiring $i_1 \neq j_1$ for the first step. We now want to see whether repeated action of the Φ_{ij} leads us again back to id. To simplify matters, we make the following assumption throughout the rest of the paper.

Assumption. *The factors r_i of the maps f_i are all equal.*

Let r denote the factor. The following statement is crucial.

Lemma 4.1. (The Φ_{ij} are expansive)
There is a neighbourhood U of id in $Sim(\mathbb{R}^d)$ such that the complement of U is mapped into itself by each Φ_{ij} :

$$\Phi_{ij}(g) \notin U \qquad \text{if} \quad g \notin U.$$

Proof. Let $f_i(x) = rA_i(x + a_i)$, thus $f_i^{-1}(x) = r^{-1}A_i^{-1}x - a_i$ for $i = 1, ..., m$, and $g(x) = sBx + b$, where A_i, B are orthogonal matrices and a_i, b are vectors in $\mathbb{R}^d$. Then

$$\Phi_{ij}(g) = f_i^{-1} \cdot g \cdot f_j = sCx + (sCa_j - a_i) + r^{-1}A_i^{-1}b$$

where C denotes the orthogonal matrix $A_i^{-1}BA_j$. If we let $c = \max_{i=1}^m |a_i|$ then $|sAa_j - a_i| \le (1+s)c$ for all i, j. Moreover, $|r^{-1}A_i^{-1}b| = |b|/r$.

Thus if $|b|(\frac{1}{r} - 1) \ge (1+s)c$, then the translational part of $\Phi_{ij}(g)$ will have length at least $|b|$. The following neighbourhood U of id now fulfils the statement:

$$U = \{g \mid \ |b| \le (1+s)c/(1-r)\}. \qquad \clubsuit$$

Since we start with $g = id$ and $\Phi_{ij}(g)$ has the same factor as g, the number s will always remain equal to 1. In other words, we need not consider the action of the Φ_{ij} on $Sim(\mathbb{R}^d)$ but only on the subspace of isometries.

To decide whether the algebraic separation condition is true, we now can disregard all mappings which fall outside the neighbourhood U. Let us first demonstrate how our method decides whether the pieces A_i of A are disjoint, and hence A is a Cantor set.

Proposition 4.2. (Self-similar sets with disjoint pieces)
The pieces A_i of A are pairwise disjoint if and only if there exists a number n_0 such that repeated application of more than n_0 automorphisms Φ_{ij} to id, with $i_1 \ne j_1$, will always lead out of the neighbourhood U.

Proof. The mappings in E can be interpreted as "neighbour maps". Let $u = (i_1, ..., i_n)$, $v = (j_1, ..., j_n)$, and $f_v = f_{j_1} ... f_{j_n}$ and $A_v = f_v(A)$, similarly for u. Then the map $f_u^{-1}f_v$ in E is the transformation which maps A_u to A_v, renormalized to the size of A. More precisely, $f_u^{-1}f_v$ maps A to the neighbouring position $f_u^{-1}f_v(A)$ which is situated to A in a geometrically similar way as A_v is situated to A_u. See [6,3].

Now if the A_i are disjoint then there is an $\epsilon > 0$ such that any two sets A_i, A_j have distance ϵ or larger. This holds also for the subsets A_u, A_v if $i_1 \ne j_1$. So the distance between A and $f_u^{-1}f_v(A)$ is at least $r^{-n}\epsilon$. Since $f_u^{-1}f_v$ is an isometry, this means that its translational part has norm at least $r^{-n}\epsilon - 2 \cdot t$ where t denotes the maximum distance of the points of A from the origin. Thus for all n greater some n_0, the maps $f_u^{-1}f_v$ fall outside the neighbourhood U of id defined above. $\clubsuit$

If we assume that the origin belongs to A – this is the case when $a_1 = 0$ – we can replace t by the diameter of A. Using the definition of U with $s = 1$, this yields the following estimate.

Corollary 4.3. (Estimate of distance between pieces)
If $a_1 = 0$ then A is a Cantor set with distance ϵ between pieces if and only if the iteration of the Φ_{ij} leads outside U after n steps, where

$$\epsilon \geq 2 \cdot r^n \cdot (\frac{c}{1-r} + \text{ diam } A). \qquad \clubsuit$$

In the following we shall study the graph G with (possibly infinite) vertex set $E \cap U$ where an edge is drawn from vertex g to vertex h for every (i,j) for which $h = \Phi_{ij}(g)$. Thus G is a directed graph which may have loops and multiple edges, and each edge has a corresponding mark (i,j). The vertex id will be considered as the root of the graph. By construction of the set E and by our lemma, every vertex can be reached from the root vertex by a finite directed path of edges. There is a very simple principle.

Principle 4.4. *For every infinite directed path of edges $e_1, e_2, ...$ starting in the root vertex id with marks $(i_1, j_1), (i_2, j_2), ...$ the two points with adresses $i_1 i_2 ...$ and $j_1 j_2 ...$ will coincide, and conversely.*

The above proposition treats the case that there is no such infinite path and in particular no directed cycle back to the root vertex id. Even if there were a single infinite directed path in G, then in general for $m = 3$ the set A will be a Cantor set while for $m = 2$ the set A must be connected, cf. [7].

Now, let us discuss the case that the vertex set $E \cap U$ is finite. This is true when the pieces are disjoint. When all the mappings $f_i^{-1} f_j$ are contained in a crystallographic group [14], $E \cap U$ is also finite [3]. We note that finite $E \cap U$ does not necessarily imply that the intersections $A_i \cap A_j$ are finite, see the example of the terdragon below. In the next section we shall also construct non-crystallographic examples where graph G is finite.

Proposition 4.5. *When $E \cap U$ is finite, it will be clearly generated by the Φ_{ij} in finitely many steps. If the graph contains a directed cycle back to id, then we have no separation, otherwise the OSC holds. Thus in the finite case the separation question is decidable in a finite number of steps.* $\clubsuit$

Proposition 4.6. *When $E \cap U$ is finite, existence of infinite directed paths in graph G is the same as existence of directed cycles. The intersections $A_i \cap A_j$ are finite if and only if there are only finitely many disjoint directed cycles in G. Any two directed cycles which share at least one edge will lead to an uncountable number of infinite directed paths in G. In particular this is true for a single directed cycle with a double edge.*

Proof. Let $e = (g, h)$ be a common edge of two cycles in G. Let w denote a path from id to g, and let w^1, w^2 be the two cycles from g over h back to g. Then for each sequence $k = k_1 k_2 k_3 ... \in \{1, 2\}^\infty$, there is a corresponding infinite path $w w^{k_1} w^{k_2} w^{k_3} ...$ $\clubsuit$

5 New Sierpinski gaskets

Clearly, the background for our algorithm comes from dynamical systems. We do not iterate a single transformation, but rather a multivalued map given by the Φ_{ij}, and the basic space is not $\mathbb{R}^d$ but the space of isometries on $\mathbb{R}^d$, with id as origin. Nevertheless, in the same way as for the Mandelbrot set of complex quadratic maps, it is decisive whether trajectories all go to infinite or to 0 or whether they stay in-between. Trajectories go to infinite as soon as they get out of the neighbourhood U.

In the following *we further specialize by requiring that all f_i have the same linear part,* $A_i = A$. Such systems of mappings have been considered by many authors in connection with self-affine tilings, cf. [18,3,14]. It turns out that in this case all Φ_{ij} act on the space of translations. We start with $g = id$, and if B is the identity matrix in the proof of the lemma, the matrix C of $\Phi_{ij}(g)$ will also be the identity matrix. So for $g(x) = x + b$ we can write

$$\Phi_{ij}(b) = a_j - a_i + r^{-1}A^{-1}b$$

indicating that our Φ_{ij} now act on $\mathbb{R}^d$. In dimension 2, it is more convenient to rA by a complex contraction factor a, and let $\lambda = 1/a$. Then $f_i(z) = a(z + a_i)$ and $f_i^{-1}(z) = \lambda z - a_i$ implies

$$\Phi_{ij}(b) = a_j - a_i + \lambda b.$$

To construct new "gaskets" like figure 2, *we now consider a self-similar set A with $m = 3$ pieces which is symmetric with respect to a 120^o rotation around the origin.* More prcisely, $f_3 = \overline{\omega} f_1 \omega$ and $f_2 = \omega f_1 \overline{\omega}$ where $\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2} = e^{i2\pi/3}$. If we put $a_1 = 1$, an easy calculation gives

$$f_i(z) = a(z + a_i) \quad \text{and} \quad f_i^{-1}(z) = \lambda z - a_i \quad \text{for } i = 1, 2, 3$$

with $a_1 = 1$, $a_2 = \omega$ and $a_3 = \overline{\omega}$.

Now we start with $b = 0$ and apply the Φ_{ij} repeatedly, disregarding all image points with modulus larger than $2/(1 - |a|)$ (cf. the definition of U). We wonder for which a (or λ) we get a finite set of iterates in U.

In the first step we require $i \neq j$ and obtain as possible $\Phi_{ij}(0)$ the six numbers $a_i - a_j$. These points form a regular hexagon centered at the origin, with two vertices $\pm i\sqrt{3}$ on the imaginary axis. These points are now taken as new b in the above formula for $\Phi_{ij}(b)$. We have to multiply one of the vertices of the hexagon by λ and add another of the hexagon vertices in order to obtain the result.

It seems natural to look for periodic solutions, so let us first check whether the result can again be a hexagon vertex. Since $|\lambda| > 1$, the point λb must be one of the 12 possible differences of hexagon vertices which lie outside the hexagon. This gives 12 possible values of λ, but because of the symmetry of the set A the numbers $\omega\lambda$ and $\overline{\omega}\lambda$ give exactly the same self-similar set as

λ. Thus we are left with four cases, and they are all well-known. $\lambda = 2$ gives the Sierpinski gasket, $\lambda = -2$ the "reversed version" shown in figure 1.

$\lambda = \frac{1}{2}(3 \pm i\sqrt{3})$ yield the right- and left-handed version of the terdragon [19] which tile the plane in a similar way as hexagons do. This is an example of the second case in proposition ? since for each vertex b of the hexagon two Φ_{ij} can be applied successfully – one leading back to b and one to its neighbour. From this fact the symbolic description of the boundaries of the pieces A_i and of the center point $A_1 \cap A_2 \cap A_3$ could be easily derived.

Next, we look for periodic solutions where only the application of two maps Φ_{ij} will lead from one hexagon vertex to another. Thus there must be hexagon vertices w, w', w'', w''', not necessarily different, such that

$$(w\lambda - w')\lambda - w'' = w''' \ .$$

Dividing by w we get

$$\lambda^2 - \eta\lambda - (\psi + \xi) = 0$$

where η, ψ and ξ belong to the set of sixth roots of unity $\{\pm 1, \pm\omega, \pm\overline{\omega}\}$. Thus

$$\lambda = \eta\left(\frac{1}{2} \pm \sqrt{\frac{1}{4} + \alpha + \beta}\right)$$

where $\eta, \alpha = \psi\eta^{-2}$ and $\beta = \xi\eta^{-2}$ are again sixth roots of unity. Because of the symmetry of A, multiplication of λ by ω or $\overline{\omega}$ will lead to the same A. Hence only $\eta = \pm 1$ need to be considered. The general solution for λ now is

$$\lambda = \pm\frac{1}{2} \pm \sqrt{\frac{1}{4} + \alpha + \beta}$$

Moreover, since the complex conjugate of λ leads to a "left-handed version" of A, it suffices to study those α, β with $Im(\alpha + \beta) \geq 0$.

There are 12 possible choices left for $\alpha + \beta$ but not all of them lead to an admissible λ. It is well known that for $r < 1/2$ the set A will be a Cantor set while for $r > 1/\sqrt{3}$ the open set condition cannot hold because the similarity dimension is larger than 2. This implies

$$\sqrt{3} \leq |\lambda| \leq 2 \ .$$

Let us now study all possible 12 sums $\alpha + \beta$ of two sixth roots of unity in the upper half-plane. On the real line we have the values $-2, -1, 0, 1$ which all give too small $|\lambda|$, and $\alpha + \beta = 2$ which yields the ordinary Sierpinski gasket and its reversed version. Obviously, periodic orbits of period 1 of a dynamical system will also be obtained when we search for orbits of period two.

Next, on the line $Im\, z = \sqrt{3}/2$, the possible values for the real part of $\alpha + \beta$ are $\pm 1/2$ and $\pm 3/2$. While $-3/2$ and $\pm 1/2$ give too small $|\lambda|$, the value

$\alpha + \beta = 3/2 + i\sqrt{3}/2$ gives an admissible solution $\lambda \approx 1.8606 + i \cdot 0.3182$ with $|\lambda| \approx 1.8876$ and rotational angle $arg\lambda \approx 9.706^o$. See figure 4. Below we prove that in this case the number of infinite directed paths in the graph G is uncountable. Thus each intersection $A_i \cap A_j$ is a Cantor set which is also indicated by the magnification.

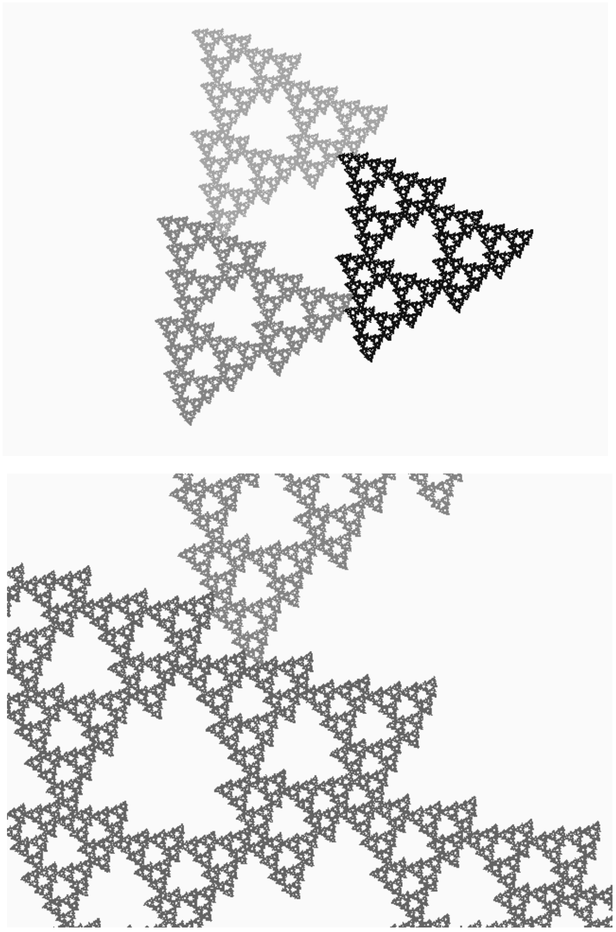

Fig. 4. A new gasket which is not p.c.f. with a magnification

Finally, on the line $Im\, z = \sqrt{3}$ we have the real part 0, which leads to the terdragon, and ± 1. For -1 we get $|\lambda| \approx 1.70$ which is a bit too small. For $\alpha + \beta = 1 + i\sqrt{3}$ we obtain figure 2. $\lambda \approx 1.8011 + i \cdot 0.6656$ has modulus about 1.92015. Thus $r \approx 0.5208$, and the angle of rotation is about 20.28^o. For $-\lambda$ we get the reversed version of figure 2, see figure 5.

For these two figures it can be easily checked that $E \cap U$ is finite, and there is exactly one cycle which alternates between the vertices of the basic hexagon and another hexagon. The point is that $\alpha = \beta$, more precisely $w = w', w'' =$

$w''' = w(1+i\sqrt{3})/2$. Hence there is one infinite directed path from id through each vertex of the basic hexagon, and since two paths marked $(i_1, j_1), (i_2, j_2)$... and $(j_1, i_1), (j_2, i_2)$... describe the same pair of adresses, these paths describe exactly three intersection points between the pieces A_i, as can be seen in the figures.

In the example of figure 4, α and β are different and can be interchanged. The same holds for w'' and w'''. From w there is an edge to $w(\lambda - 1)$, and from there we have two edges, to $w \cdot (1 + i\sqrt{3})$, and another back to w. This explains the Cantor set structure of $A_i \cap A_j$.

The methods presented here can easily be generalized in different ways. We can consider orbits of period three and higher, determined by the roots of polynomials of degree larger than 2. We can study 4 or more maps instead of 3, and we can also drop the assumption of symmetry. In each of these cases, calculations become more intricate, and a lot of new and interesting examples of fractals can be determined.

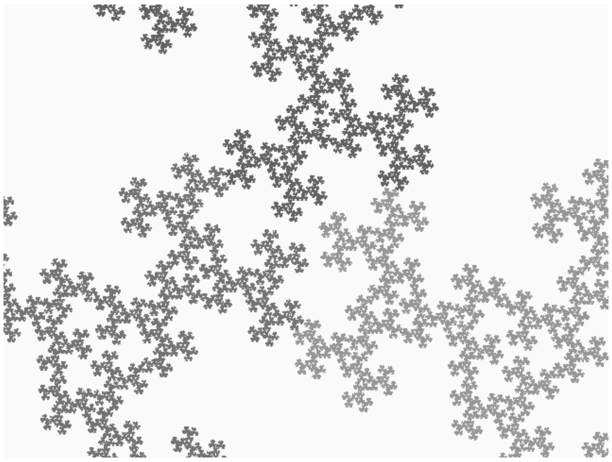

Fig. 5. Magnification of the reversed version of fig. 2.

6 The exponent of interior metric

In this section, we determine shortest paths between points in figure 2. First we study the boundary or post-critical set of A. It will turn out it consists of the 12 points labelled a_i, b_i, c_i, d_i with $i = 1, 2, 3$ in figure 6. We have $f_i(a_i) = b_i, f_i(b_i) = c_i, f(c_i) = d_i$ and $f_{i+1}(d_i) = a_{i+1}$ so that the points form a 12-cycle in counterclockwise order. Here + is mod 3, that is, 3+1=1. The two points $f_i(c_{i+1})$ and $f_{i+1}(a_i)$ coincide – this is where pieces touch each other.

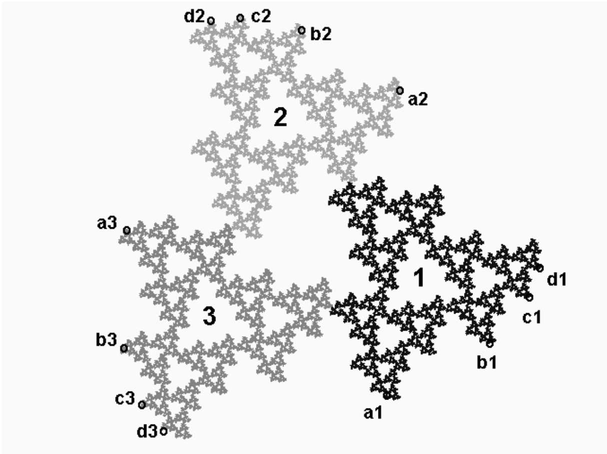

Fig. 6. The boundary of our gasket.

In [8] it was shown that the system of shortest paths between boundary points is itself a system of self-similar sets, determined by a system of set equations. Because of symmetry, we need only consider 16 different shortest paths $P(x_i, y_{i+1})$, with $x, y \in \{a, b, c, d\}$. Each of these passes through one "touching point", and hence is the union of two shortest paths between boundary points in the pieces, that is, of two other shortest paths scaled down by r. Because of the open set condition, the system of these paths has a similarity dimension η. The Hausdorff measure of dimension η of shortest paths defines an interior metric on A [8]. In particular, if the length of a path P is ℓ, then the length of a scaled-down version of that path is $r^\eta \ell$.

Let $\gamma = r^\eta$, and let us introduce the notation xy for the shortest distance between x_i and y_{i+1} (more precisely, $xy = \mu^\eta(P(x_i, y_{i+1}))$ [8]) where x, y denote any two symbols in $\{a, b, c, d\}$. Then we immediately obtain the following equations which reflect the division of any path into two smaller paths.

$$aa = \gamma cd + \gamma^2 dd,\ ab = \gamma(cd + aa),\ ac = \gamma(cd + ab),\ bd = 2\gamma ac,$$

$$bc = \gamma(ac + ab),\ cc = \gamma(bc + ab),\ cd = \gamma(bc + ac),\ dd = \gamma(cc + ac).$$

To fix a multiplicative constant, let $dd = 1$. Now all variables can be expressed in terms of γ and cd.

$$aa = \gamma cd + \gamma^2,\ ab = (\gamma + \gamma^2)cd + \gamma^3,\ ac = (\gamma + \gamma^2 + \gamma^3)cd + \gamma^4,$$

$$bd = 2\gamma ac,\ \ , bc = (2\gamma^2 + 2\gamma^3 + \gamma^4)cd + (\gamma^4 + \gamma^5),$$

$$cd = (\gamma^2 + 3\gamma^3 + 3\gamma^4 + \gamma^5)cd + (2\gamma^5 + \gamma^6).$$

Now cd is obtained as a rational function of γ . Substituting in

$$1 = \gamma(cc + ac) = (\gamma^2 + 2\gamma^3 + 4\gamma^4 + 2\gamma^5 + \gamma^6)cd + (2\gamma^5 + \gamma^6 + \gamma^7)$$

we get the following result.

Proposition 6.1. (Exponent of interior metric for figure 2) *γ is the real root of the polynomial* $\gamma^7+\gamma^6+3\gamma^5+3\gamma^4+3\gamma^3+\gamma^2-1$. *Thus* $\gamma \approx 0.51141$. *The exponent of interior metric is* $\eta = \frac{\log \gamma}{\log r} \approx 1.028$.

This is not very far from 1. Note that the Hausdorff dimension of A is $\frac{\log 3}{-\log r} \approx 1.684$, somewhat larger than the dimension $\frac{\log 3}{\log 2} \approx 1.585$ of the Sierpinski gasket.

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