Existence, bifurcation, and stability of profiles for classical and non-classical shock waves

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Abstract. This paper surveys the authors' recent results on viscous shock waves in PDE systems of conservation laws with non-convexity and non-strict hyperbolicity. Particular attention is paid to the physical model of magnetohydrodynamics. The plan of the paper is as follows. Sections 1 and 2 introduce the classes of systems and the classes of shock waves we consider and recall how profiles for small-amplitude shocks are constructed via center manifold analyses of a corresponding system of ODEs. Section 3 describes the global picture, i. e., large-amplitude shock waves, for the case of magnetohydrodynamics, first the solution set of the Rankine-Hugoniot jump conditions, then a heteroclinic bifurcation occurring in the ODE system for the profiles. Section 4 presents a method for the numerical identification of heteroclinic manifolds, which is applied in Sections 5 and 6 to the case of magnetohydrodynamics. The numerical treatment confirms and details the analytical findings and, more notably, extends them considerably; in particular, it allows to study the existence / non-existence of profiles and the aforementioned heteroclinic bifurcation globally. Section 7 dicusses the stability of viscous shock waves; the important nonuniformity of the vanishing viscosity limit for, in particular, non-classical MHD shock waves is *not* addressed in this paper.

1 Classification of shock waves

Let U be an open subset of $\mathbb{R}^n$ and $g, f : U \to \mathbb{R}^n$ smooth functions such that g maps U diffeomorphic onto its image, while $(Dg(u))^{-1}Df(u)$ is $\mathbb{R}$-diagonalizable at every $u \in U$. Consider the hyperbolic system of conservation laws

$$g(u)_t + f(u)_x = 0, \tag{1}$$

and a non-characteristic *inviscid shock wave*

$$u(x,t) = \begin{cases} u^- : x - st < 0, \\ u^+ : x - st > 0. \end{cases} \tag{2}$$

associated with (1), i. e., the triple $(u^-, u^+, s) \in U \times U \times \mathbb{R}$ with $u^- \neq u^+$ satisfies the Rankine-Hugoniot conditions

$$-s(g(u^+) - g(u^-)) + (f(u^+) - f(u^-)) = 0$$

and s is not an eigenvalue of $Df(u^-)$ nor of $Df(u^+)$.

To classify such objects, introduce, for arbitrary $(u,s) \in U \times \mathbb{R}$, the spaces

$$R^-(u,s) = \sum_{\lambda<s} \ker(Df(u) - \lambda Dg(u)), \quad R^+(u,s) = \sum_{\lambda>s} \ker(Df(u) - \lambda Dg(u)).$$

The shock wave is called *Laxian*, or *classical*, if the linearized Rankine–Hugoniot conditions

$$\begin{aligned} & g(u^+) - g(u^-))\bar{\sigma}' \\ & + (Df(u^+) - sDg(u^+))\bar{u}_+^+ - (Df(u^-) - sDg(u^-))\bar{u}_-^- \\ = & -(Df(u^+) - sDg(u^+))\bar{u}_-^+ + (Df(u^-) - sDg(u^-))\bar{u}_+^- \end{aligned} \tag{3}$$

have a unique solution $(\bar{u}_-^-, \bar{u}_+^+, \bar{\sigma}') \in R^-(u^-,s) \times R^+(u^+,s) \times \mathbb{R}$ for any $(\bar{u}_+^-, \bar{u}_-^+) \in R^+(u^-,s) \times R^-(u^+,s)$. Generally, let

$$l = \dim\, R^-(u^-,s) + \dim\, R^+(u^+,s) + 1$$

and

$$r = \dim\, (R^-(u^-,s) + R^+(u^+,s) + \mathbb{R}(u^+ - u^-))$$

be the number of unknowns and the rank, respectively, of (3). Let

$$\underline{\kappa} = l - r \geq 0, \qquad \overline{\kappa} = n - r \geq 0$$

denote the degrees of *under-* resp. *overdeterminacy* of this linear algebraic system. A shock wave with $\underline{\kappa} > 0$ (and $\overline{\kappa} = 0$) is called *(purely) undercompressive*; a shock wave with $\overline{\kappa} > 0$ (and $\underline{\kappa} = 0$) is called *(purely) overcompressive.* For any shock wave, call the ordered pair

$(\underline{\kappa}, \overline{\kappa})$ the *algebraic type* of the shock wave

and the integer

$$\kappa = \overline{\kappa} - \underline{\kappa} + 1$$

its *multiplicity*. Letting n^-, n^+ denote the dimensions of the spaces of "incoming" modes to the left and right of the shock wave, respectively, i. e.,

$$n^- = \dim\, R^+(u^-,s), n^+ = \dim\, R^-(u^+,s),$$

we have

$$\kappa = n - l + 1 = n - (2n - (n^- + n^+)) = n^- + n^+ - n. \tag{4}$$

Together with the 'inviscid' system (1), we consider the 'viscous' system

$$g(u)_t + f(u)_x = (B(u)u_x)_x \tag{5}$$

with some appropriate *viscosity* $B : U \to I\!R^{n\times n}$. A traveling wave solution $u(t,x) = \phi(x - st)$ of (5) corresponding to a given inviscid shock wave (2) is called its *viscous profile*. Writing $q \equiv -sg(u^{\pm}) + f(u^{\pm})$, such profile technically is a heteroclinic orbit of

$$B(\phi)\phi' = f(\phi) - sg(\phi) - q, \tag{6}$$

with end states

$$\phi(\pm\infty) = u^{\pm}.$$

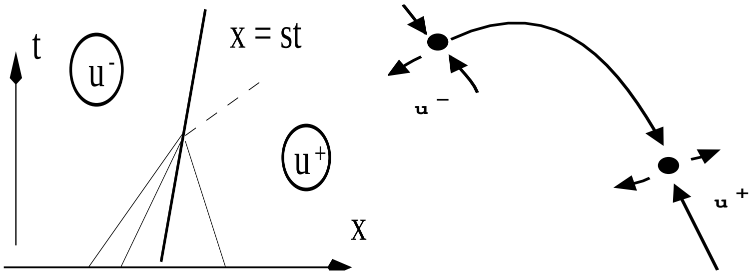

Fig. 1. Laxian shock, $(\underline{\kappa}, \overline{\kappa}) = (0, 0)$. Example with $n = 2$, $\kappa = 1 = k$.

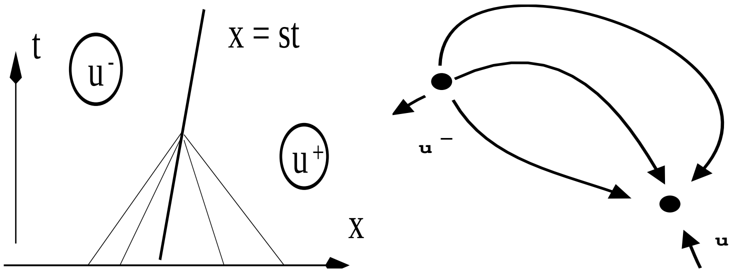

Fig. 2. Overcompressive shock, $\overline{\kappa} > 0$. Example with $n = 2$, $(\underline{\kappa}, \overline{\kappa}) = (0, 1), \kappa = 2 = k$.

Assuming for a moment that B has full rank n and the rest points $u^{\pm}$ of (6) are hyperbolic, we let

$$k^- = \dim W^u(u^-),\ k^+ = \dim W^s(u^+)$$

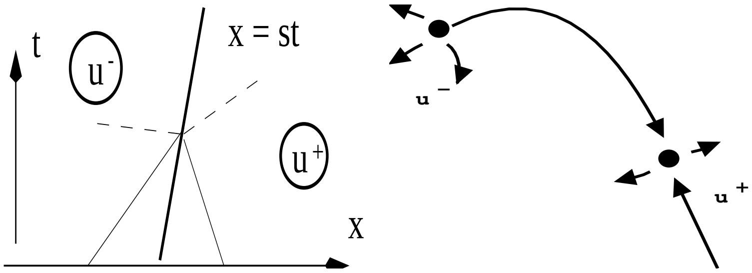

Fig. 3. Undercompressive shock, $\underline{\kappa} > 0$. Example with $n = 2$, $(\underline{\kappa}, \overline{\kappa}) = (1, 0)$, $\kappa = 0 = k$.

denote the dimensions of the unstable manifold of (6) at u^- and the dimension of the stable manifold of (6) at u^+, respectively, and define the *index* of the viscous profile ϕ as

$$k = k^- + k^+ - n.$$

Under certain conditions on B, the dimensions of $W^u(u^-)$ and $W^s(u^+)$ are equal to those of $R^+(u^-, s)$ and $R^-(u^+, s)$, i. e., $k^- = n^-, k^+ = n^+$ so that, by (4),

$$\text{multiplicity } \kappa \text{ of the shock} = \text{index } k \text{ of its profile.} \tag{7}$$

This holds, e. g., for $B = I$, the identity matrix, in which case $R^+(u^-, s)$, $R^-(u^+, s)$ are the tangent spaces, at u^-, u^+, of $W^u(u^-), W^s(u^+)$.) In each of Figures 1,2,3, the left picture shows an inviscid shock wave together with characteristics ($\dot{x} = \lambda(u^\pm)$, $\lambda(u^\pm)$ eigenvalues of $Df(u^\pm)$, while the right picture sketches a corresponding phase portrait for the profile ODE (6).

2 Profiles for small-amplitude shock waves

We henceforth restrict attention to *symmetric*, canonically splitting systems, i. e., we assume that $G \equiv Dg$, $F \equiv Df$, and B are symmetric matrices with G positive definite and B positive semidefinite, and (5) decomposes as

$$\begin{aligned} &\begin{pmatrix} G_{11}(v,w) & G_{12}(v,w) \\ G_{21}(v,w) & G_{22}(v,w) \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix}_t + \begin{pmatrix} F_{11}(v,w) & F_{12}(v,w) \\ F_{21}(v,w) & F_{22}(v,w) \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix}_x \\ &\qquad = \left(\begin{pmatrix} 0 & 0 \\ 0 & \tilde{B}(v,w) \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix}_x \right)_x \end{aligned} \tag{8}$$

with $\tilde{B}$ positive definite.

Theorem 1. *Consider a simple mode* (λ, r), *i. e.,*

$$(F(u) - \lambda(u)G(u))r(u) = 0, \tag{9}$$

where λ is real-valued and the vector field $r \neq 0$ is unique up to a scalar factor. Writing

$$F - \lambda G \equiv A \equiv \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

assume that at some state u_*,

$$A_{11}(u_*) \text{ is invertible.} \tag{10}$$

Then any small λ-shock near u_* (i. e., any shock with end states $u^-, u^+ \approx u_*$ satisfying $\lambda(u^-) > s > \lambda(u^+)$) has a viscous profile w. r. t. the viscosity B if and only if it satisfies the strict version $(E)_s$ of Liu's entropy condition [L1]. In particular, if the mode is convex (cf. (33) below), every sufficiently small λ-shock has a profile.

Proof of Theorem 2.1. Analogously to (8), we decompose (6) in the form

$$0 = f_1(v,w) - sg_1(v,w) - q_1 \tag{11}$$
$$\tilde{B}(v,w)w' = f_2(v,w) - sg_2(v,w) - q_2. \tag{12}$$

Consider (6) resp. (11),(12) for $u = (v,w)$ near $u_* = (v_*, w_*)$ and (q,s) near (q_*, s_*) with

$$s_* = \lambda(u_*) \text{ and } q_* = f(u_*) - s_* g(u_*).$$

Assumption (10) implies that (11) can be solved locally for v as

$$v = V(w,q,s), \tag{13}$$

i. e.,

$$0 = f_1(V(w,q,s),w) - sg_1(V(w,q,s),w) - q_1. \tag{14}$$

Plugging V into (12), we obtain the reduced system

$$\hat{B}(w,q,s)w' = h(w,q,s), \tag{15}$$

with

$$\begin{aligned} \hat{B}(w,q,s) &\equiv \tilde{B}(V(w,q,s),w), \\ h(w,q,s) &\equiv f_2(V(w,q,s),w) - sg_2(V(w,q,s),w) - q_2. \end{aligned} \tag{16}$$

More precisely, (13) and (15) together are equivalent to (6). We claim now that any two states sufficiently close to u_* that form a λ-shock are located on a one-dimensional invariant manifold C of (6). To see this, note first that

$$\begin{aligned} & D_w h|_{(q,s)=(q_*,s_*)} r_2 = 0 \\ & \quad \Leftrightarrow Ar = 0 \quad \text{with} \quad r = (r_1, r_2), r_1 = -(A_{11})^{-1} A_{12} r_2. \end{aligned} \tag{17}$$

Equivalence (17) follows from

$$0 = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$$
$$\Leftrightarrow r_1 = -(A_{11})^{-1} A_{12} r_2 \quad \text{and} \quad (-A_{21}(A_{11})^{-1} A_{12} + A_{22}) r_2 = 0$$

and

$$D_w h = A_{21} D_w V + A_{22} = -A_{21}(A_{11})^{-1} A_{12} + A_{22},$$

the latter identity coming from (14) as $0 = A_{11} D_w V + A_{12}$.

Note now that Assumption (10) implies that the w-component r_2 of the eigenvector $r = (r_1, r_2)$ does not vanish. This means that $D_w h(w_*, q_*, s_*)$, and thus $\hat{B}^{-1}(w_*, q_*, s_*) D_w h(w_*, q_*, s_*)$, have a one-dimensional kernel spanned by r_2. As $\hat{B}$ and h_w are symmetric and $\hat{B}$ is positive, $\hat{B}^{-1} h_w$ cannot have any purely imaginary eigenvalue other than 0. Applying the Center Manifold Theorem to system (15) as augmented by the further equations

$$q' = 0, s' = 0, \tag{18}$$

we see that the augmented system has, near (w_*, q_*, s_*), a center manifold with 1-dimensional w-fibres ((q, s)-sections). The left and right states u^-, u^+ of any small shock under consideration are rest points of (6). As any center manifold contains locally all rest points of the flow to which it belongs, there is precisely one fibre C (—lifting via (14), we immediately view C as lying in u-space—) that contains u^- and u^+. It is now easy to see that the open segment of C between u^- and u^+ is the desired profile if and only if there exists no other fixed point between these two. This is however equivalent to Liu's condition in its strict version $(E)_s$: For any u located between u^- and u^+ on the Hugoniot locus

$$\mathcal{H}(u^-) = \{u \in U \,:\, \exists s = s(u, u^-) : \, s(u^-, u)(g(u) - g(u^-)) = f(u) - f(u^-)\},$$

the strict inequality $s(u, u^-) < s(u^+, u^-)$ holds. Cf. [Fre2] for details. Theorem 2.1 is considered proved.

A number of important systems from continuum mechanics are of the form (8). Instances are the equations of compressible viscous, heat-conducting fluids as well as those of compressible magnetohydrodynamics in various variants corresponding to the simultaneous presence or non-presence of dissipative mechanisms associated with viscosity, heat conductivity, and electrical resistivity, when written in entropy variables. Cf. [Kw] for the identification of this class of systems and that of the mentioned physical systems as examples. Notice that in most—though not all—of these examples, the existence of viscous profiles, even for shocks of large amplitude, has been shown through *ad hoc* considerations [Gi,CS]. The purpose of the above part of the present section is to demonstrate the use of the Center Manifold Theorem in the context of degenerate viscosity which is in fact quite similar to the nondegenerate case[MP].

We now turn to non-classical shock waves. Non-classical shock waves of small amplitude arise near *umbilic points*, i. e., points near which modes, see (9), change multiplicity. For the construction of viscous profiles for small non-classical shock waves, one considers center manifolds as above, but with fibers C of dimension higher than 1. To illustrate what one can obtain in this way, we now focus on the concrete system that constitutes the primary object of our more detailed investigations. Plane waves in viscous, resistive, heat-conductive magnetohydrodynamics (MHD) satisfy the equations

$$\begin{aligned}
\rho_t + (\rho v)_x &= 0 \\
(\rho v)_t + (\rho v^2 + p + \frac{1}{2}|\mathbf{b}|^2)_x &= \zeta v_{xx} \\
(\rho \mathbf{w})_t + (\rho v \mathbf{w} - a\mathbf{b})_x &= \mu \mathbf{w}_{xx} \\
\mathbf{b}_t + (v\mathbf{b} - a\mathbf{w})_x &= \nu \mathbf{b}_{xx} \\
\mathcal{E}_t + (v(\mathcal{E} + p + \frac{1}{2}(|\mathbf{b}|^2 - a^2)) - a\mathbf{w}\cdot\mathbf{b})_x &= \kappa\theta_{xx} + \zeta(v v_x)_x \\
&\quad + \mu(\mathbf{w}\cdot\mathbf{w}_x)_x + \nu(\mathbf{b}\cdot\mathbf{b}_x)_x,
\end{aligned} \tag{19}$$

where $v, \mathbf{w}$ and $a, \mathbf{b}$ are the longitudinal and transverse components of the fluid's velocity $\mathbf{V} = (v, \mathbf{w}) = (v, w_1, w_2)$ and the magnetic field $\mathbf{B} = (a, \mathbf{b}) = (a, b_1, b_2)$, respectively, ($a \equiv \mathit{const}$ as $\mathit{div}\ \mathbf{B} = 0$) and $\mathcal{E} = \rho(\frac{1}{2}|\mathbf{V}|^2 + \epsilon) + \frac{1}{2}|\mathbf{B}|^2$, is the density of total energy. The variables $\rho, p, \theta, \epsilon$, describing density, pressure, temperature, and internal energy of the fluid, are intrinsically related with each other through the equation of state $\epsilon = \epsilon(\tau, \eta)$ and the identities $\rho = \tau^{-1},\ p = -\epsilon_\tau(\tau, \eta),\ \theta = \epsilon_\eta(\tau, \eta)$, where τ denotes the specific volume and η the entropy of the fluid. The internal energy ϵ is required to satisfy the conditions $-\epsilon_\tau > 0$, $\epsilon_\eta > 0$, $D^2\epsilon > 0$, $-\epsilon_{\tau\eta} > 0$, $-\epsilon_{\tau\tau\tau} > 0$; the first two of these requirements amount to the positivity of pressure and temperature, the third to the concavity of entropy η as a function of τ and e, and the fourth and fifth are known as "Weyl's conditions." The two dissipation coefficients $\mu \geq 0$ and $\zeta \geq 0$ correspond to the intrinsic viscosity of the fluid; more precisely, $\mu = \zeta_1, \zeta = \zeta_2 + \frac{4}{3}\zeta_1$ with $\zeta_1, \zeta_2 \geq 0$ the first and second viscosity coefficients of the fluid. The two remaining coefficients $\nu \geq 0$ and $\kappa \geq 0$ denote the electrical resistivity and the thermal conductivity of the fluid. We recall (e. g. from [KuLi]) some basic properties of *ideal MHD*, i. e., Eqs. (19) with $\zeta = \mu = \nu = \kappa = 0$.

The seven characteristic speeds $\lambda_{-3} \leq \lambda_{-2} \leq \lambda_{-1} \leq \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3$ of this 7×7 hyperbolic system of conservation laws are of the form

$$\lambda_0 = v, \quad \lambda_{\pm 1} = v \pm c_-, \quad \lambda_{\pm 2} = v \pm c_A, \quad \lambda_{\pm 3} = v \pm c_+ \tag{20}$$

with the fast and slow magnetoacoustic speeds $c_+ \geq c_- \geq 0$ given by $c_\pm^2 = \frac{1}{2}[(c_s^2 + \rho^{-1}(a^2 + b^2)) \pm \sqrt{(c_s^2 + \rho^{-1}(a^2 + b^2))^2 - 4c_s^2\rho^{-1}a^2}]$ (where c_s is the sound speed, $c_s^2 = \pi_\rho(\rho,\eta)$ with $\pi(\rho,\eta) = p = -\epsilon_\tau(\tau,\eta))$ and the Alfvén speed $c_A \geq 0$ by $c_A^2 = \rho^{-1}a^2$. We assume henceforth that $a \neq 0$. Obviously, $0 < c_- < c_A < c_+$ if $\mathbf{b} \neq 0$. For $\mathbf{b} = 0$ however,

$$\begin{array}{lll} 0 < c_- = c_A < c_+ & \text{if} & \Delta_a > 0, \\ 0 < c_- = c_A = c_+ & \text{if} & \Delta_a = 0, \\ 0 < c_- < c_A = c_+ & \text{if} & \Delta_a < 0, \end{array} \tag{21}$$

with $\Delta_a = \rho c_s^2 - a^2$. Typically, all three cases in (21) occur, with Δ_a vanishing along a smooth manifold which separates its own complement into two open sets where $\Delta_a > 0$ and $\Delta_a < 0$, respectively. E. g., $\Delta_a = \gamma p - a^2$ for a perfect gas $\epsilon(\tau,\eta) = c_v \exp(\eta/c_v)\tau^{1-\gamma}$. For shock waves, with, say w. l. o. g., $s = 0$, the Rankine-Hugoniot conditions require u^- and u^+ to satisfy

$$f(u) = q \tag{22}$$

with the same value of the relative flux q. For $q \in Q$, the set of regular values of the mapping f, Eqs. (22) have up to four solutions u_0, u_1, u_2, u_3 satisfying

$$\begin{array}{r} 0 < \pm\lambda_{\mp 3}(u_0) \\ \pm\lambda_{\mp 3}(u_1) < 0 < \pm\lambda_{\mp 2}(u_1) \\ \pm\lambda_{\mp 2}(u_2) < 0 < \pm\lambda_{\mp 1}(u_2) \\ \pm\lambda_{\mp 1}(u_3) < 0 < \pm\lambda_0 \ \ (u_3) \end{array} \tag{23}$$

With the two cases in (23) differing only by a direction reversal $x \mapsto -x$, we restrict attention to first one (upper signs) without loss of generality. The four states u_0, u_1, u_2, u_3 combinatorially allow for various inviscid shock waves (2), namely the twelve species $u^- = u_i, \quad u^+ = u_j, \quad i,j \in \{0,1,2,3\}, i \neq j$, which are briefly referred to as being of species $i \to j$. As entropy increases with the index, i. e., $\eta(u_0) < \eta(u_1) < \eta(u_2) < \eta(u_3)$, only shocks of species $i \to j$ with $i < j$ are thermodynamically possible. One distinguishes between the classical shocks of species $0 \to 1, 2 \to 3$ which are associated with the fast and slow magnetoacoustic modes c_+, c_-, respectively, and the non-classical or "intermediate" shocks of species $0 \to 2$, $1 \to 3$, $0 \to 3$, and $1 \to 2$.

Theorem 2. *Consider an arbitrary state u_* with transverse magnetic field $\mathbf{b}_* = 0$, and an arbitrary array $\delta = (\zeta, \mu, \nu, \kappa)$ of positive dissipation coefficients. Then for any $\varepsilon > 0$, there exist shock waves, with $|u^\pm - u_*| < \varepsilon$, of types $0 \to 1$, $0 \to 2$, and $1 \to 2$ (if $c_- < c_A = c_+$ at u_*), of types $2 \to 3$, $1 \to 3$, and $1 \to 2$ (if $c_- = c_A < c_+$ at u_*), or of types $0 \to 1$, $2 \to 3$, $0 \to 2$, $1 \to 3$, $0 \to 3$, and $1 \to 2$ (if $c_- = c_A = c_+$ at u_*), which possess a viscous profile with respect to the prescribed δ. More precisely, in each of these cases, shocks of type $i \to j$ have a $(j-i)$-parameter family of profiles if $j - i > 1$, and 2 profiles if $(i,j) = (1,2)$.*

The proof via considerations about the flow on 2- respectively 3-dimensional center manifolds can be found in [Fre1].

We conclude the section by connecting the MHD specific distinction of species "$i \to j$" with the general classification introduced in Section 1. It suffices to note that shocks of species $0 \to 1$, $2 \to 3$ have algebraic type $(0,0)$, shocks of species $0 \to 2$, $1 \to 3$ have type $(1,0)$, shocks of species $0 \to 3$ type $(2,0)$, and shocks of species $1 \to 2$ are of type $(1,1)$. Thus all intermediate MHD shock waves are overcompressive.

3 Bifurcation analysis for MHD shock waves

In this section we collect first results of a bifurcation analysis for the Rankine–Hugoniot relations (22) in magnetohydrodynamics, and then recall a conjecture on a related global bifurcation occurring for viscous profiles of MHD shock waves. Attention is now restricted to a perfect gas, $p = R\rho\theta$, $\epsilon = c_v\theta$.

Equivariance and rescaling considerations entitle restriction, w. l. o. g., to the three-parameter family of cases

$$\begin{aligned} \rho v &= 1, \\ v + R\theta/v + |\mathbf{b}|^2/2 &= j, \\ \mathbf{w} - \mathbf{b} &= \mathbf{0}, \\ vb_1 - w_1 &= c, \\ vb_2 - w_2 &= 0, \\ \tfrac{v^2+|\mathbf{w}|^2}{2} + (c_v + R)\theta + v|\mathbf{b}|^2 - \mathbf{w}\cdot\mathbf{b} &= e. \end{aligned} \tag{24}$$

At first consider the case $c > 0$. It is well known that there can be up to four distinct states that solve (24). The two fast states u_0, u_1 satisfy $v_0 > v_1 > 1$ while the slow states u_2, u_3 satisfy $v_3 < v_2 < 1$. The typical configuration in the $b_1 v$–plane is displayed in Figure 4. The subsequent lemma gives a more precise statement on the existence of *physical* solutions $u_0, \ldots, u_3$, i.e. states with positive pressure.

Lemma 3. *For the adiabatic coefficient $\gamma = 1 + R/c_v$, let $\bar{c}$ be the smallest (positive) solution of*

$$c^{2/3} - \sqrt{\gamma+1}\,\frac{2-\gamma}{3\gamma}c = \frac{2}{3\gamma}.$$

For each $c \in (0, \bar{c})$ there is a non-empty bounded open set $\mathcal{I}_c \subset (0,\infty)^2$ such that for each $(j, e) \in \mathcal{I}_c$ there exist four distinct physical states $u_0, \ldots, u_3$ satisfying (24).

Lemma 3 is illustrated in the right picture of Figure 4. The curve Γ_c^s denotes the set of all points $(j, e) \in \mathbb{R}^2$ such that (24) has exactly one fast solution denoted by $u_{2=3}$. For points $(j, e) \in \mathbb{R}^2$ to the left of Γ_c^s there are no slow solutions, for $(j, e) \in \mathbb{R}^2$ to the right of Γ_c^s there are two slow solutions u_2, u_3. Γ_c^f marks the analogous partition of the je–plane for the fast solutions.

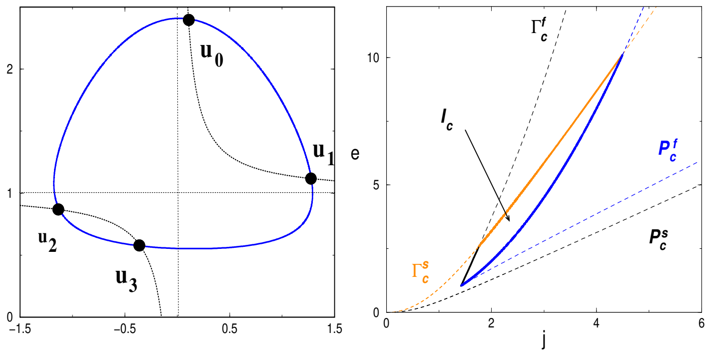

Fig. 4. Null clines of (24) in $b_1 v$–plane and the set $\mathcal{I}_c$.

The curve P_c^f consists of an upper and a lower part, ending in a cusp-type singularity for $(j,e) = (1+\frac{3}{2}c^{2/3}, \frac{1}{2}+\frac{3}{2}c^{2/3}+\frac{3}{2}c^{4/3})$. It identifies, for the upper (lower) part, the loci where the pressure $p_0 = \rho_0 R\theta_0$ $(p_1 = \rho_1 R\theta_1)$ vanishes and changes sign. Parts of these three curves –marked with solid lines in Figure 4–form the boundary of $\mathcal{I}_c$.
The singular case $c = 0$ is of particular interest. The states u_1, u_2 degenerate to a onedimensional curve of states solving (24).

Lemma 4. *Let $c = 0$ and $\gamma = 1 + R/c_v$. There is a non-empty bounded open set $\mathcal{I}_0 \subset (0,\infty)^2$ such that $(j,e) \in \mathcal{I}_0$ if and only if*

(i) there are two physical states $u_0, u_3 \in I\!R^7$ with $v_0 > 1 > v_3$ solving (24), and
(ii) there is a set A of physical states solving (24) given by

$$A = \left\{(\rho, v, \mathbf{w}, \mathbf{b}, \theta) \,|\, \rho = v = 1, |\mathbf{w}| = |\mathbf{b}| = r, \theta = \tfrac{1}{R}(j - \tfrac{r^2}{2} - 1))\right\},$$
$$r = \sqrt{2\gamma j - 2(\gamma-1)e - \gamma - 1}.$$

A is called the Alfvén circle.

Although Lemmas 3.1, 3.2—to our knowledge— cannot be found in the literature, we stress that they just refine and complement findings that trace back to the early work of Germain or Kulikovskii and Liubimov [Ge,KuLi]. For a proof of (a more detailed statement of) Lemma 3 and similar results we refer to [FreR3]. The profile ODE (6) in the MHD case, here rather (15), becomes

$$\begin{aligned}
\zeta\dot{v} &= v + p + \tfrac{1}{2} \mid \mathbf{b} \mid^2 - j,\\
\mu\dot{\mathbf{w}} &= \mathbf{w} - \mathbf{b},\\
\nu\dot{\mathbf{b}} &= v\mathbf{b} - \mathbf{w} + (c,0)^T,\\
\kappa\dot{\theta} &= c_v\theta - \tfrac{1}{2}(|\mathbf{w}|^2 - 2\mathbf{b}\cdot\mathbf{w} + v|\mathbf{b}|^2) - \tfrac{v^2}{2} + jv + \mathbf{b}\cdot(c,0)^T - e.
\end{aligned}$$
$$(\boldsymbol{\Sigma^6})$$

Obviously solutions of (24) are rest points of Σ^6. Conley and Smoller showed that the (Laxian) shock waves $u_0 \to u_1$ and $u_2 \to u_3$ admit a viscous profile [CS] for all $\delta \in (0,\infty)^4$ and all q such that the associated rest points exist. The situation for the intermediate waves is more complicated. The known (analytical and numerical) results from literature support the following conjecture:

There exists a threshold $\omega^* = \omega^*(q, \mu/\zeta, \kappa/\zeta) > 0$ such that the following holds for all $c \in (0,\bar{c})$, $(j,e) \in \mathcal{I}_c$, and $\delta = (\nu,\zeta,\mu,\kappa) \in (0,\infty)^4$: If $\nu/\zeta > \omega^*$, then all intermediate shocks (for the given q) have viscous profiles (for the given δ). Conversely, if $\nu/\zeta < \omega^*$, then no intermediate shock wave has a profile.

A proof of this conjecture for small μ and κ, following [KuLi], can be found in [FreSzm].

4 Numerical Identification of Heteroclinic Manifolds

Motivated by the dynamics of the ODE–system $\boldsymbol{\Sigma^6}$ we are (mainly) interested in viscous profiles of shock waves that appear as several–parameter families of heteroclinic orbits. In this chapter we review a direct method to approximate general heteroclinic manifolds that has been presented in [FreR1,FreR2]. Although one key ingredient is strongly connected to the analysis of conservation laws, the method can technically be viewed as a straightforward generalization of Beyn's work for single connecting orbits [Be].
To describe the method in its general context, consider any vector field $H \in C^2(I\!R^n, I\!R^n)$, $n \in \mathbf{N}$, with two hyperbolic zeros u^- and u^+. For the ODE

$$\dot{\phi} = H(\phi), \tag{25}$$

consider a non-empty family Φ of orbits connecting the rest points u^- and u^+:

$$\Phi = \{\phi \,|\, \dot{\phi} = H(\phi) \text{ and } \phi(\pm\infty) = u^{\pm}\}.$$

Furthermore, we assume that the intersection of the unstable manifold of u^- and the stable manifold of u^+, given by $\{\phi(x) \,|\, \phi \in \Phi, x \in (-\infty,\infty)\}$, is a smooth manifold of dimension d for some $d \in \{1,\ldots,m\}$. In order to parametrize Φ define a mapping

$$\Omega : \Phi \to I\!R^n$$

by

$$\Omega(\phi) \equiv \int_{I\!R} A(x,\phi(x))(\phi(x)-\phi_*(x))dx,$$

with some appropriate function $A : I\!R \times I\!R^n \to I\!R^{n\times n}$ and ϕ_* either an element of Φ or given by

$$\phi_* = \begin{cases} u^- : x < 0, \\ u^+ : x > 0. \end{cases}$$

Note that, in the case $A = \mathrm{Id}$, $\Omega(\phi)$ is the relative mass of ϕ with respect to the reference object ϕ_*, a quantity with a particular natural meaning in the case of viscous profiles. The subsequent assumption means that Ω is a chart of Φ.

Assumption 5. *The mapping Ω is injective and the range $S = \Omega(\Phi)$ is a d–dimensional manifold in $I\!R^n$ allowing for a global chart $\mathbf{P} : S \to T \equiv \mathbf{P}(S) \subset I\!R^d$.*
The corresponding parameterization of Φ as $\{\phi^\tau\}_{\tau\in T}$ with ϕ^τ defined by

$$\mathbf{P}\Omega(\phi^\tau) = \tau, \tau \in T,$$

is differentiable.

For a detailed discussion of the parametrization by relative masses and the validity of Assumption 5, in particular for conservation laws, we refer to [FreR1]. Let us note that in this field the validity of Assumption 5 is a necessary condition for time–asymptotic stability (in a certain well–defined sense) of Φ as a solution of the associated PDE. Cf. partly also Section 7 of this paper.
By Assumption 5 the problem

$$\dot{\phi}^\tau = H(\phi^\tau), \quad \phi^\tau(\mp\infty) = u^\mp, \quad \mathbf{P}\int_{I\!R} \left(\phi^\tau - \phi_*\right) = \tau, \tag{26}$$

has a unique solution $\phi^\tau \in C^1(I\!R)$, for $\tau \in T$.
Following the work of Beyn [Be] we restrict the problem (26) to a bounded interval $I = [X_-^\tau, X_+^\tau], X_-^\tau < 0 < X_+^\tau$. The approximate solution $\phi_I^\tau \in C^1(I)$ then is supposed to fulfil

$$\dot{\phi}_I^\tau = H(\phi_I^\tau) \text{ in } I, \quad b_\mp(\phi_I^\tau(X_\mp^\tau)) = 0, \quad \mathbf{P}\int_I (\phi_I^\tau - \phi_*) = \tau. \tag{27}$$

Here the functions $b_\mp$ denote asymptotic boundary conditions, for example the spectral projections associated with the unstable/stable part of the spectrum of $DH(u^\mp)$.
Following the analysis of Beyn, as presented in [Be], it is possible to derive a

rigorous convergence estimate for the error $\|\phi^\tau - \phi_I^\tau\|_{C^1(I)}$ if $|X_-^\tau|, X_+^\tau$ tend to ∞. For this sake let us assume that the d–parameter family Φ is nondegenerate in the following sense: The number $d+n$ (d dimension of the heteroclinic manifold) is given by the sum of the dimensions of the unstable subspace of $DH(u^-)$ and the stable subspace of $DH(u^+)$. Furthermore for each $\tau \in T$ we have

$$\dot{y} = DH(\phi^\tau)y,\ y(\mp\infty) = 0 \Leftrightarrow y \in \operatorname{span}\left\{\frac{\partial \phi^\tau}{\partial \tau_1}, \dots, \frac{\partial \phi^\tau}{\partial \tau_d}\right\}.$$

Under these assumptions (and some technical requirements on $b_\mp$) we can prove

Theorem 6. *For each $\tau \in T$ there is a $\tilde{X}^\tau > 0$ such that for any $I = [X_-^\tau, X_+^\tau]$ with $|X_-^\tau|, X_+^\tau > \tilde{X}^\tau$ we have:*

(i) There is a $\delta > 0$ such that there exists a unique solution $\phi_I^\tau \in C^1(I)$ of the truncated problem (27) with $\|\phi_I^\tau - \phi^\tau\|_{C^1(I)} \le \delta$.
(ii) There is a constant $C = C(\tau) > 0$ such that:

$$\|\phi_I^\tau - \phi^\tau\|_{C^1(I)} \le C|I| \exp\left(-\min\{\lambda^- X_-^\tau, -\lambda^+ X_+^\tau\}\right), \qquad (28)$$

where λ^- (λ^+) are given by the minimal absolute value of the real parts of the unstable (stable) eigenvalues of DH at u^- (u^+).

For a detailed proof we refer to Section 4 in [FreR2]. Note that it cannot be expected that Theorem 6, in particular (28), holds uniformly for all $\tau \in T$. This issue will be further discussed in Section 6 below.

5 Numerical Study of the Heteroclinic Bifurcation in MHD

In this section we report on systematic investigations into the MHD profiles ODE system $\boldsymbol{\Sigma^6}$ using the method described in Section 4. The results illustrate dynamically interesting scenarios, in particular in regimes that could so far not be, and seem hard to be, covered analytically.
We consider the global bifurcation scenario of $\boldsymbol{\Sigma^6}$ that has been described in Section 3. While the validity of this conjecture is only proven for small values of μ and κ, numerical results that we will present in this section support the conjecture that the scenario remains *globally* true, i. e., for all $\kappa, \mu > 0$ and $(j,e) \in I_c, c \in (0,\bar{c})$. In [FreR1] we presented two methods to decide whether the global bifurcation takes place or not. We will not go into detail but mention that the methods rely on the refined conjecture that the bifurcation can be completely analyzed in an four–dimensional linear subspace E that is invariant with respect to the flow of $\boldsymbol{\Sigma^6}$. Figures 5 show some results: the bifurcation ratio ω^* for fixed $\mu = 0.01$ and different values of κ in the left

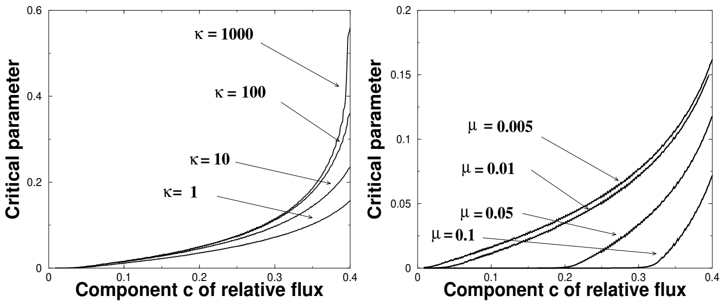

Fig. 5. Critical parameter ω^* versus c

picture, the bifurcation ratio ω^* for fixed $\kappa = 1$ and different values of μ in the right picture. ω^* was calculated for a series of values for c, j, e such that $c \in (0, \bar{c})$ and $(j_c, e_c) \in I_c$.

We observe that ω^* vanishes for $c \to 0$ which coincides with the fact that the counterparts of the intermediate shock waves in the degenerate case $c = 0$ — the switch–on/off shock waves — have profiles for all values of δ and $(j, e) \in I_0$ (cf. Section 6 for the orbit structure in the case $c = 0$). However, it is not true, that the bifurcation parameter is uniformly bounded from above for all $\mu, \kappa > 0$ and all $c \in (0, \bar{c}), (j, e) \in I_c$ as certain partial earlier results of Wu in [W] may suggest. Figure 6 shows that the bifurcation ratio ω^* tends to ∞ as the heat conductivity κ tends to ∞, for $\mu > 0$ and q fixed.

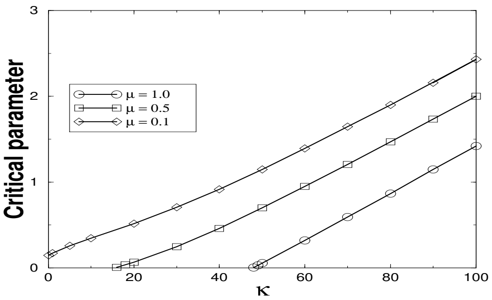

Fig. 6. Critical parameter ω^* versus κ.

Now we illustrate the bifurcation by a series of computations with the method described in Section 4.
Before starting let us mention some details of the implementation. The truncated problem (27) can be solved with any kind of BVP–solver, in principle. We actually use the code COLNEW [BaA] which relies on a variable step–

size collocation method.
Concerning the approximation of the higher dimensional heteroclinic manifolds in $\boldsymbol{\Sigma}^{\mathbf{6}}$ we will focus on the manifolds of type $u_0 \to u_2$ and $u_1 \to u_3$ and proceed as follows. Define the set T in (27) by

$$T = \{(0,0,0,\tau_1,\tau_2,0) \,|\, \tau_1 \in I\!R, \tau_2 \in (-\bar{\tau},\bar{\tau})\}, \quad \bar{\tau} \equiv \bar{\tau}(q,\delta) \equiv |\int_{I\!R} b_2(x)\,dx|. \tag{29}$$

Here the function b_2 refers to the (already computed) b_2–component of one of the orbits of type $u_1 \to u_2$. Note that τ_2 is associated to the component b_2 and that b_2 vanishes for all rest points such that the integral in (29) is finite. Now, we approximate the bounded manifolds completely when freezing the first τ–component τ_1, lets say $\tau_1 = 0$, and continuing in the parameter τ_2 starting with $\tau_2 = 0$.

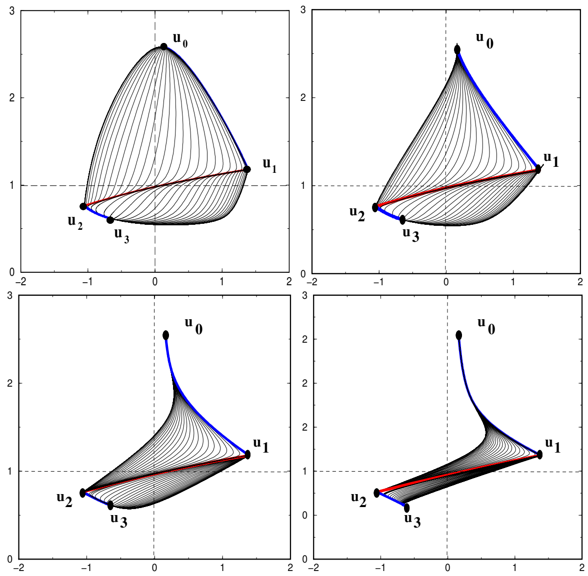

Fig. 7. Projection to b_1v–plane for $\omega = 7.5, 1.0, 0.25, 0.06$.

For the above–mentioned illustration, we fix the transverse fluid viscosity μ, heat conductivity κ and some $c \in (0,\bar{c})$, $(j,e) \in I_c$, to be specific:

$$\mu = 0.01, \quad \kappa = 1, \quad c = 0.25, \quad (j,e) = (2.68, 4.23). \tag{30}$$

By variation of the remaining free parameter, the ratio $\omega = \nu/\zeta$, we observe the global bifurcation. The numerically calculated orbits of all types except $u_0 \to u_3$ are displayed in Figure 7 as projections to the b_1v–plane. We picked

out the configurations for $\omega = 7.5, 1.0, 0.25, 0.06$. For the 3D–version in $b_1 b_2 v$–space (cf. Figure 12), the colour of the surfaces which have been constructed from the orbit families of type $u_0 \to u_2$ and $u_1 \to u_3$ represent the values of the temperature θ on the corresponding heteroclinic manifold. The figures have been processed using the visualization platform GRAPE [g]. An MPEG–video showing the dynamics in more detail can be found at [w].
For the chosen set of parameters the critical value ω^* is approximately 0.0492. The graphs in Figure 8 display the situation for $\omega = \omega^*$ where only the single orbits $u_0 \to u_1$, $u_2 \to u_3$, and $u_1 \to u_2$ exist and $\omega = 0.02 < \omega_*$ where also $u_1 \to u_2$ is broken.

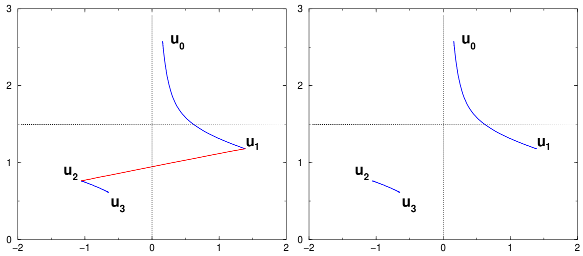

Fig. 8. Projection to $b_1 v$–plane for $\omega = 0.0492, 0.02$.

6 Boundary Cases: (Almost-) Symmetry and Fast-Slow Dynamics

We now discuss an important special case, along with situations where our method, though reliable and robust, reaches its limitations. If the component c of the relative flux q is strictly bigger than zero the system $\boldsymbol{\Sigma^6}$ has up to four isolated rest points $u_0, \dots, u_3$ located in the invariant subspace E. For $c = 0$, the rest points u_0, u_3 persist in E while u_1, u_2 degenerate to a circle of rest points (Alfvén circle, cf. Section 3). In particular, for $c = 0$ the solution set of $\boldsymbol{\Sigma^6}$ in the $b_1 b_2 v$–space is rotationally symmetric with respect to the v–axis.

In a series of pictures (Figures 9, 10) we show the orbit types $u_0 \to u_1, u_0 \to u_2, u_1 \to u_2, u_1 \to u_3, u_2 \to u_3$ for different values of c. We fixed $\nu = 0.5$, $\zeta = 1$ and μ, κ, j, e as in (30). For $c \to 0$, the phase portrait comes closer and closer to being symmetric with respect to the $b = 0$ axis, obviously. For $c = 0$, the rest points on the Alfvén circle are not hyperbolic, but the connecting orbits reach it along the exponentially stable part of the center–stable manifold.
We now comment on a numerical problem, arising at two levels. Firstly, for

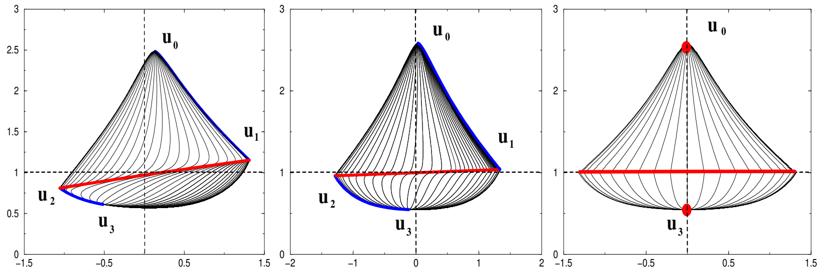

Fig. 9. Viscous profiles of type $u_0 \to u_1$, $u_0 \to u_2$, $u_1 \to u_2$, $u_2 \to u_3$ and $u_1 \to u_3$ for $c = 0.25, 0.05$ and viscous profiles connecting u_0/u_3 with the Alfvén circle for $c = 0$, projected to the b_1v–plane.

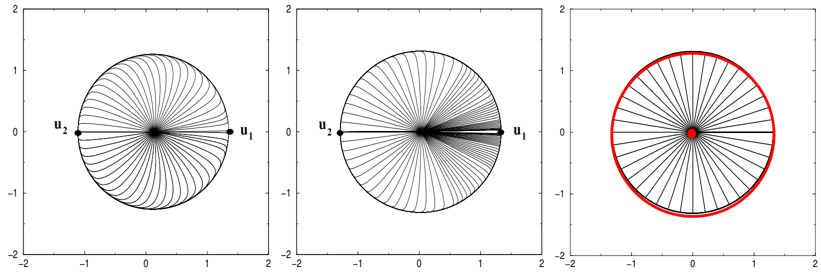

Fig. 10. The heteroclinic manifold $u_0 \to u_2$ for $c = 0.25, 0.05$ and viscous profiles connecting u_0 with the Alfvén circle for $c = 0$, projected to the b_1b_2–plane.

fixed q, the approximation of the heteroclinic manifolds of higher dimension, say $u_1 \to u_3$, is done by a continuation precedure starting with the orbit lying in the linear subspace E (for parameter $\tau = (0, \tau_2) = (0, 0)$) and ending with some orbit close to the boundary of the heterclinic manifold ($\tau = (0, \bar{\tau})$). During this continuation process, the truncation interval $[X_-^\tau, X_+^\tau]$ is enlarged, indeed exponentially, to capture the orbits in a satisfying manner. This is due to the fact that the orbits near the boundary of the manifold are closer and closer to the further rest point u_2.
Secondly, if we consider the intervals $[X_-^\tau, X_+^\tau]$, $\tau_2 \in [0, \bar{\tau})$ for decreasing c, we observe that the interval length grows even "more". This happens since in the limit $c \to 0$, the orbits of type $u_1 \to u_2$ become slower and slower, before in the limit case $c = 0$ (the closure of) their union degenerates to the Alfvén circle.
We exemplify these effects by the results of a numerical experiment, illustrated in Figure 11. We have plotted, against τ_2, the minimal intervals $[X_-^\tau, X_+^\tau]$, such that the criterion $|\phi_\tau(X_\mp) - u_\mp| < \varepsilon$ is satisfied, for a given small tolerance $\varepsilon > 0$. As τ_2 increases and c decreases, these intervals grow to an extent which indicates that the method needs a refinement if one wishes to resolve these regimes more efficiently.

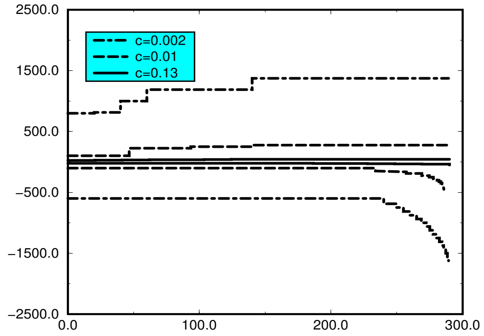

Fig. 11. Interval end points versus the continuation parameter τ_2.

7 Stability of viscous shock waves

This section discusses an analytical result on the stability of small-amplitude viscous Laxian shock waves associated with possibly nonconvex modes.

Theorem 7. *Consider a system of viscous hyperbolic conservation laws*

$$u_t + f(u)_x = \varepsilon u_{xx}, \tag{31}$$

$f \in \mathrm{C}^3(\mathbb{R}^n, \mathbb{R}^n)$, $\varepsilon > 0$. *Let* $u_* \in \mathbb{R}^n$ *be a fixed reference state and let* $\phi : \mathbb{R} \to \mathbb{R}^n$ *denote the profile of a Laxian shock wave, near* u_*, *associated with a simple eigenvalue* λ *of* f', *i.e., the states* $u^\pm$ *are close to* u_* *and satisfy* $\lambda(u^-) > s > \lambda(u^+)$.

Let X *be the completion of* $\{\bar{u} \in \mathrm{C}_0^\infty(\mathbb{R}, \mathbb{R}^n),\ \int_{-\infty}^{\infty} \bar{u}(x)dx = 0\}$ *under the norm* $\|\bar{u}\| = \|\bar{u}\|_{\mathrm{L}_1} + \|\bar{u}\|_{\mathrm{H}^1}$.

There exist positive constants ϵ_0, β_0 *such that if* ϕ *satisfies* $|u^+ - u^-| < \epsilon_0$ *and* $u_0 - \phi \in X$ *satisfies* $\|u_0 - \phi\| < \beta_0$, *then the solution* $u(x,t)$ *to* (31) *with data* $u(\cdot, 0) = u_0$ *exists for all times* $t > 0$ *and has*

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}} |u(x,t) - \phi(x - st)| = 0. \tag{32}$$

Briefly speaking, profiles of small-amplitude shock waves are *time-asymptotically stable.*

The result described in Theorem 7 was obtained by Goodman [Go] under the additional assumption that the eigenvalue λ be *convex*, i.e.

$$r \cdot \nabla\lambda \neq 0 \text{ where } r = \ker(f' - \lambda I). \tag{33}$$

For the general case of possibly non-convex modes, it is due to Fries [Fri1].

The stability of shock profiles had been investigated by Il'in and Oleinik [IO] for the case of a scalar equation with strictly convex flux function. Sattinger [Sa] used spectral methods to prove a stability result for travelling wave solutions of general parabolic systems which implies stability of Laxian shock profiles under perturbations of exponential decay. Goodman's proof for systems used the energy method, an approach that was introduced into the context of viscous hyperbolic conservation laws by Goodman and independently by Matsumura and Nishihara [MN1]. The result of Goodman was extended by Liu [L2,L3] and by Szepessy and Xin [SzeX] to the case of non-zero mass perturbations—still using the assumption of strict convexity. In [L3] the energy method was no longer involved, which enabled the derivation of pointwise decay-estimates. The stability of shock profiles for the non-convex scalar equation was shown by Matsumura and Nishihara [MN2] via a weigthed energy method; Gardner, Jones and Kapitula [JGKp] used a spectral approach, while Freistühler and Serre [FreSe] employed contractivity of the semigroup to establish global L_1 stability.

Recently, spectral methods have been developed to investigate the instability and stability of travelling waves for large classes of systems [GaZ,ZH]. In contrast with their enormous importance both for qualitative insight as well as for computational access, these new methods have so far not been able to provide alternative proofs for facts like the one established in Theorem 7.

We return to Theorem 7 by remarking that it has recently been extended to the non-zero mass case [Fri2]; in this note, we stay however away from the subtleties of this much more complicated case. We outline the proof of the Theorem 7 in the version it has been stated above.

We consider the solution u of (31) with initial data u_0 "close" to the profile ϕ, about which latter we assume without loss of generality that $s = 0$. Subtracting, we have

$$(u-\phi)_t + (f(u) - f(\phi))_x - \varepsilon(u-\phi)_{xx} = 0$$

i.e.

$$(u-\phi)_t + (f'(\phi)(u-\phi))_x - \varepsilon(u-\phi)_{xx} + (Q(\phi, u-\phi))_x = 0.$$

where Q is given by the Taylor expansion $f(\phi + u - \phi) - f(\phi) = f'(\phi)(u - \phi) + Q(\phi, u-\phi)$. It is thus natural to diagonalize f'. For simplicity in presentation we assume that *all* eigenvalues of f' are simple. Thus we find smooth matrix valued functions L, R such that $Lf'R = \Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$ where the eigenvalues λ_k are such that $\mathrm{sign}\lambda_k = \mathrm{sign}(k-p)$ for $k \neq p$ and $\lambda_p = \lambda - s$, i.e. $\lambda_p(u_-) > 0 > \lambda_p(u_+)$. Introducing the integrated variable $U(x,t) := \int_{-\infty}^{x} u(\xi,t) - \phi(\xi)d\xi$ we have $U(\cdot, 0) \in \mathrm{H}^2(\mathbb{R})$ and obtain the *integrated equation* for U

$$U_t + f'(\phi)U_x - \varepsilon U_{xx} + Q(\phi, U_x) = 0 \tag{34}$$

Changing to characteristic coordinates $V := LU$ we obtain a diagonalized version of (34):

$$\begin{aligned} V_t + \Lambda(\phi)V_x - \varepsilon V_{xx} + LQ(\phi, RV_x) \\ + \Lambda(\phi)LR_xV - 2\varepsilon LR_xV_x - \varepsilon LR_{xx}V = 0. \end{aligned}$$

Multiplying this equation by $V^{\mathsf{T}}(x,t)W(x)$, where $W = \mathrm{diag}(w_1, \dots, w_n)$ is a weight matrix yet to be defined, and integrating $\int_{-\infty}^{\infty} dx$, we arrive at

$$\int_{-\infty}^{\infty} \frac{1}{2}(V^{\mathsf{T}}WV)_t - \frac{1}{2}V^{\mathsf{T}}[W\Lambda + \varepsilon W_x]_x V + \varepsilon V_x W V_x + \mathit{error\ terms} = 0, \tag{35}$$

where integration by parts was used.

Under the assumption of a weak shock and by an appropriate choice of the weights w_i (see below) it is possible to obtain $-\frac{1}{2}[W\Lambda + \varepsilon W_x]_x \geq c|\phi_x|\mathrm{I}$ and $W \geq \mathrm{I}$ and thus

$$\frac{1}{2}\frac{\partial}{\partial t}\|V\|^2_{\mathrm{L}_2} + c|\phi_x|\|V\|^2_{\mathrm{L}_2} + c\varepsilon\|\sqrt{W}V_x\|^2_{\mathrm{L}_2} + \mathit{error\ terms} = 0. \tag{36}$$

With the assumption of a weak shock (small $|u_+ - u_-|$) and small perturbations (small β_0) we can estimate the error terms against the others. Integrating $\int_0^T dt$ and returning to the original variable U gives

$$\|U(\cdot,T)\|^2_{\mathrm{L}_2} + \int_0^T \|U_x(\cdot,t)\|^2_{\mathrm{L}_2} dt \leq C\|U(\cdot,0)\|^2_{\mathrm{L}_2}, \tag{37}$$

where $T > 0$ denotes any time until which U exists with $U(\cdot,t)$, $0 \leq t \leq T$, lying in the appropriate spaces, and $C > 0$ is a constant independent of T. With this estimate it is easy to obtain the same with H^2 in place of L_2: To do so we differentiate (34) once or twice with respect to x and multiply by U_x or U_{xx} respectively. Crude estimation of terms involving h and Q then gives

$$\|U_x(\cdot,T)\|^2_{\mathrm{L}_2} + \int_0^T \|U_{xx}(\cdot,t)\|^2_{\mathrm{L}_2} dt \leq C\Big(\|U_x(\cdot,0)\|^2_{\mathrm{L}_2} + \int_0^T \|U_x(\cdot,t)\|^2_{\mathrm{L}_2} dt\Big)$$

and

$$\|U_{xx}(\cdot,T)\|^2_{\mathrm{L}_2} \leq C(\|U_{xx}(\cdot,0)\|^2_{\mathrm{L}_2} + \int_0^T \|U_x(\cdot,t)\|^2_{\mathrm{L}_2} + \|U_{xx}(\cdot,t)\|^2_{\mathrm{L}_2} dt).$$

Combining this with the above we obtain

$$\|U(\cdot,T)\|^2_{\mathrm{H}^2} + \int_0^T \|U_x(\cdot,t)\|^2_{\mathrm{L}_2} dt \ \leq \ C\|U(\cdot,0)\|^2_{\mathrm{H}^2}.$$

Since (34) is a uniformly parabolic system, one has a short time existence result for U. Thus the above a-priori estimate gives global existence of U, thus of u, and finally $\lim_{t\to\infty}\sup_{x\in I\!R}|u(x,t)-\phi(x-st)| = \lim_{t\to\infty}\sup_{x\in I\!R}|U_x(x,t)| = 0$.

To summarize, the two main difficulties in the whole argumentation are the choice of the weights w_k to obtain $-\frac{1}{2}[W\Lambda+\varepsilon W_x]_x \geq c|\phi_x|\mathrm{I}$ and the estimate of the error terms (especially the coupling terms involving WLR_x) to pass from (35) via (36) to (37). As regards the weights, for each k family with $k\neq p$, the choice $w_k(x)=\exp(-\int_{-\infty}^x C|\phi_x|\mathrm{sign}\lambda_k)$ gives $-\frac{1}{2}[w_k\lambda_k+\varepsilon w_{k,x}]_x \geq c|\phi_x|$ if C is sufficiently large and the shock is sufficiently weak (i.e. $|u_+-u_-|$ sufficiently small). In the strictly convex case now, the choice $w_p\equiv 1$ yields the same property for $k=p$, allowing at the same time for straightforward treatment of the error terms [Go].

For a non-convex mode, i.e. when $\lambda_p(\phi(x))$ is *not* decreasing in x, it is still possible to obtain $-\frac{1}{2}[w_p\lambda_p+\varepsilon w_{p,x}]_x \geq c|\phi_x|$ through appropriate choice of w_p. But in this case it is non-trivial to estimate the error terms. The difficulty is easily seen from the following heuristic investigation of some properties of the weight w_p: When restricting to the scalar case, a typical non-convex flux would be $f(u)=u^3$. In this case we see that $\varepsilon\phi_x = h(\phi)$ is of the order ϵ^3 ($\epsilon := |u_+-u_-|$). Since λ_p is then of the order ϵ^2 we see from $-\frac{1}{2}[w_p\lambda_p+\varepsilon w_{p,x}] \sim c\int|\phi_x|dx \sim \epsilon$ that w_p is of the order ϵ^{-1}. The point (of the difficulty) now is that the weight retains this order when considering a coupled system. Since the weight w_p appears in the error terms, it is not immediately clear that they can be estimated for ϵ sufficiently small. So far for the strategy and the difficulties arising in the proof of Theorem 7. For any further details we must refer the reader to [Fri1,Fri2].

We conclude by remarking that non-convex modes occur naturally in physical systems. An example is again provided by magnetohydrodynamics. Restricting the twodimensonal variables $\mathbf{b},\mathbf{w}$, i. e., the transverse components of the magnetic field and the velocity, in (19) to a fixed line—this corresponds to restricting the full magnetic field and velocity vectors to a fixed plane: the so-called coplanar case—, one arrives at the 5 by 5 system

$$\begin{aligned}
\rho_t + (\rho v)_x &= 0\\
(\rho v)_t + (\rho v^2 + p + \frac{1}{2}|b|^2)_x &= \zeta v_{xx}\\
(\rho w)_t + (\rho v w - ab)_x &= \mu w_{xx}\\
b_t + (vb - aw)_x &= \nu b_{xx}\\
\mathcal{E}_t + (v(\mathcal{E} + p + \frac{1}{2}(|b|^2 - a^2)) - awb)_x &= \kappa\theta_{xx}\\
&\quad + \zeta(vv_x)_x + \mu(ww_x)_x + \nu(bb_x)_x
\end{aligned} \tag{38}$$

The *coplanar system* (38) retains five of the seven modes (20) of (19). The rotational modes $\lambda_{-2},\lambda_{+2}$ being absent in (38), points (with $b=0$) that, considered as states of the full system, have eigenvalues of multiplicity 2 by $\lambda_{\pm 2}$ coinciding with either $\lambda_{\pm 1}$ or $\lambda_{\pm 3}$, are now, for (38), points of strict hyperbolicity, and small-amplitude intermediate shocks near these points are now Laxian shocks associated with the non-convex (!) simple modes $\lambda_{\pm 1}$ or $\lambda_{\pm 3}$.

Still, Theorem 7 does of course not readily apply, as it requires the "artificial" viscosity $B = \varepsilon I$. Also, despite the conclusive study of a model problem [FreL], it is not clear how to re-proceed from the nonconvex coplanar problem to the non-classical non-coplanar problem. Work on both issues is in progress.

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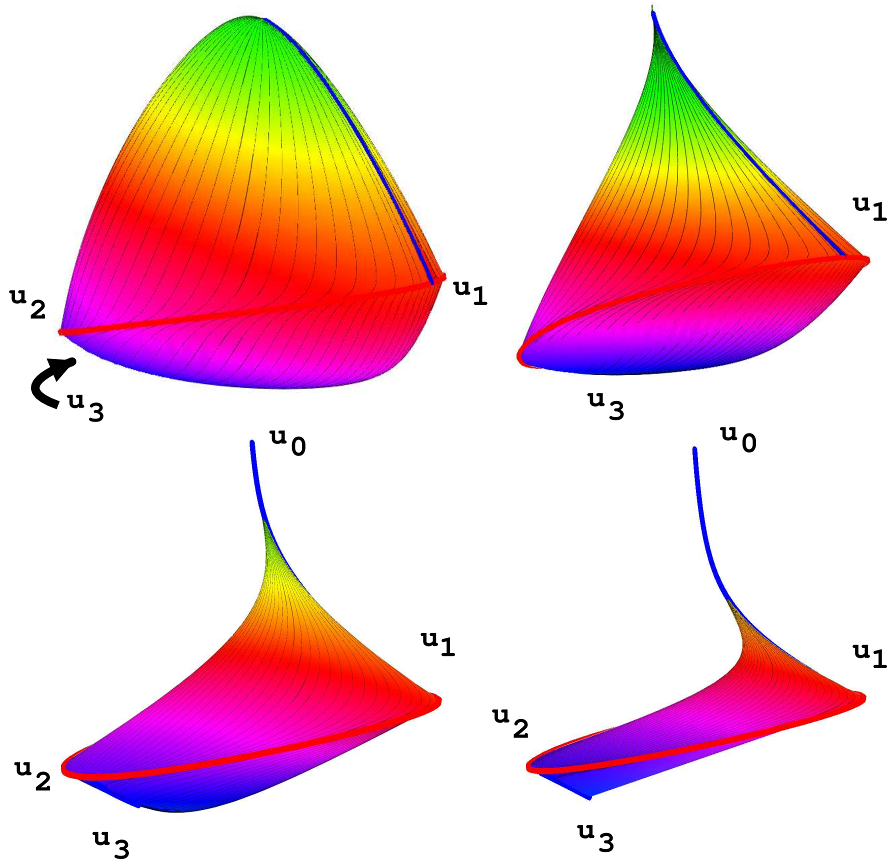

Fig. 12. Projection to $b_1 b_2 v$–space for $\omega = 7.5, 1.0, 0.25, 0.06$.