Forced symmetry breaking and relative periodic orbits

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Abstract. We would like to discuss some aspects of forced symmetry breaking in equivariant systems. This topic is of significance for applications since in real life systems symmetries occur only as approximations. This is in particular relevant for the applications of genericity theories. There are certain dynamical features (heteroclinic cycles) which are generic within the class of symmetric systems, however they are of high codimension in the context of general dynamical systems. Therefore one might expect that forced symmetry breaking destroys such behavior. However it has been observed that weak perturbations of symmetric systems can introduce new complicated dynamical properties. Therefore it is interesting to study the influence of weak symmetry breaking from a purely mathematical point of view as well as from an applied point of view. In this paper we focus on the mathematical part and we try to describe some of our ideas in geometrical terms.

1 The setting and basic results

1.1 Basic definitions

In this paper we study perturbations of differential equation which break the symmetry of these equations. So we start with an equivariant vectorfield $f : \mathbb{R}^n \to \mathbb{R}^n$ (or more generally $f : M \to TM$) and the corresponding equation

$$\dot{x} = f(x). \tag{1}$$

Equivariance means that the trajectories are mapped onto trajectories under the linear action of a Lie group G. Precise definitions will be given shortly. If small terms of the form $\varepsilon h(x)$ perturb this equation and h transforms the same way as f but only with respect to a subgroup $H < G$, then we call this *forced symmetry breaking.* Besides some basic notions from group theory which are needed to describe the setting of equivariant systems we will need some terminology to describe the geometry induced by group actions. It turns out that this geometry provides a lot of structure which makes it easier to understand such complicated dynamical systems.

Let us begin with a vector space V and a compact Lie group G acting on V. The first and simplest notions are the *isotropy subgroup of a point* $x \in V$

$$G_x = \left\{ g \in G \;\middle|\; gx = x \right\}$$

and the *group orbit*

$$Gx = \left\{ gx \;\middle|\; g \in G \right\}.$$

Since we are interested in special time orbits, which intersect a group orbit either for all $t \in \mathbb{R}$ or for a certain subset $\mathbb{T} \subset \mathbb{R}$, we find it convenient to work on the *orbit space* V/G. This orbit space is the topological space obtained from identifying group orbits to points. Since we need to do analysis on such orbit spaces we have to introduce a topology, which is the finest one, such that the natural projection, ie the map

$$\pi : V \to V/G : x \mapsto Gx$$

is continuous. Note, that although we have a smooth structure on V and although we consider smooth actions, the orbit space need not be a smooth manifold. In fact in many cases V/G is not a smooth manifold. The simplest such case is the $\mathbb{Z}_2$ action on $\mathbb{R}$, which is given by $x \mapsto -x$. The orbit space can be represented by the set of points $\left\{ x \in \mathbb{R} \;\middle|\; x \geq 0 \right\}$. Obviously the set of nonnegative reals is not a manifold and we need some additional structure to account for this bad behavior.

In Figure 1 we indicate an orbit space and its relation to dynamics.

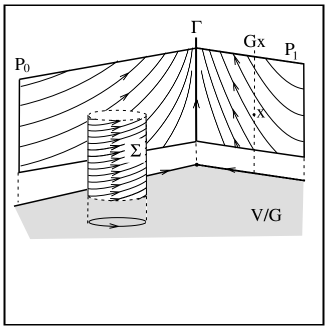

Fig. 1. A schematic view of an orbit space.

An important observation which is fundamental for all to come is, that the isotropy subgroups of points on one group orbit are conjugate, ie if $y = Gx$, then

$$G_y = gG_xg^{-1}.$$

The proof of this fact is elementary and left to the reader, however this observation allows us to speak of the orbit type: for a point x we say its orbit type $\tau(x)$ is the equivalence class of groups which are conjugate to G_x, ie $\tau(x) = [G_x] = \left\{ gG_xg^{-1} \;\middle|\; g \in G \right\}$. Since the above observation tells us, that the orbit type is constant along group orbits, we can introduce a map $\bar{\tau}$ from V/G to the set of orbit types, ie to the set of conjugacy classes of subgroups of G, by mapping $Gx \to \tau(x) = [G_x]$. So we can decompose the topological space V/G into sets of points of the same orbit type. So let $S_\tau = \left\{ Gx \in V/G \;\middle|\; \bar{\tau}(Gx) = \tau \right\}$.

T *he connected component of S_τ is a smooth manifold. The set of all connected components of all S_τ, where τ runs through the set of conjugacy classes of subgroups is a stratification of V/G into smooth strata.*

We call this construction the stratification of V and of V/G into isotropy types. Note, that sometimes in the literature the orbit type G/G_x is used instead of the isotropy type G_x. SCHWARZ [28] shows that one can define a smooth structure on V/G such that the projection of a smooth equivariant vectorfields is smooth.

Another important concept, which is in some sense dual to the isotropy subgroup is the fixed point space, ie given a subgroup $H < G$, we write

$$\mathrm{Fix}(H) = \left\{ v \in V \;\middle|\; hv = v \text{ for all } h \in H \right\}.$$

Closely related to the fixed point space is the normalizer of H, which is denoted by $\mathrm{N}(()H)$. The normalizer acts naturally on $\mathrm{Fix}(H)$.

Now we want to begin to study dynamics in terms of these geometric notions. We consider equivariant ODE's, ie equations of the type

$$\dot{x} = f(x), \tag{2}$$

where f is equivariant, ie $f(gx) = gf(x)$ for all $g \in G$. The orbit space allows to define *relative objects* for such equations in a natural manner. Observe, that due to the above observation there is a way to project vectorfield on the orbit space and to define a flow for such a projected field. Let us write $\pi^* f$ for the image of f under this mapping. Details of such a construction may be found in [17,18,23,26,25,27,28]. Let $\pi^*\Phi$ be the flow on V/G which is the projection of the flow Φ on V. Observe that equivariance implies that strata are flow invariant. This is the first observation in terms of geometric restrictions for equivariant flows.

The first example of a relative object is the relative equilibrium, which we define as follows.

Definition 1. A relative equilibrium is the preimage $\pi^{-1}(y_0)$ of an equilibrium $y_0 \in V/G$ with respect to the flow $\pi^*\Phi$.

In the literature one finds other descriptions of relative equilibria, for example that a relative equilibrium is a group orbit which is invariant under the flow, ie that the time orbit is part of the group orbit. If we take a one-parameter subgroup of G which corresponds to a time orbit on Gx, we observe a steady state solution in this particular moving frame. We shall see later that this construction of relative objects applies also to more complicated objects like relative periodic orbits or other relative invariant sets.

Examples for relative equilibria are equilibria, of course and more generally periodic motions of bodies moving with constant angular velocity around one of their axis of symmetry.

1.2 Forced symmetry breaking: a first look

Forced symmetry breaking means the systematic study of equivariant equations which are perturbed slightly so that they loose part of their symmetry. Assume f is a G-equivariant vectorfield and $H \subset G$ is a closed subgroup. Consider an H-equivariant vectorfield g and the resulting equation on V

$$\dot{x} = f(x) + \varepsilon g(x).$$

Of course changing the symmetry group of an equation has severe consequences. If V is a sum of a certain number of irreducible representations of G it will decompose into irreducible representations of H. In general this decomposition is rather complicated. Even if V is an absolutely irreducible representation we might find many representations of H which interact in a complicated manner. As an example we recall the $\mathbf{SO}(3)$ situation. Irreducible representations are in any odd dimension and for each dimension we find up to equivalence precisely one such representation. If H equals $\mathbf{SO}(2)$ its irreducible representations are of dimension at most two and if V has dimension $2\ell + 1$ it decomposes into $\ell + 1$ representations of $\mathbf{SO}(2)$. As a consequence local bifurcation points split into several bifurcation points and we may find many connecting orbits, most of them will approach the underlying steady state when ε tends to 0. Depending on the precise nature of the perturbation g this splitting can be different and we get a complicated zoo of bifurcating solutions. Even in simple cases there is (or seems to be) no hope to get a complete picture of what is going on. However, if we refrain from describing the global situation by following all local bifurcations we might be able to get some reasonable statements. So let us assume that we are in a reasonable distance d to all local bifurcation points, where reasonable means that this distance should be in a relation to the size of the perturbation. In fact we need a bit more. To describe the necessary ingredients, let us look at the stability of a relative equilibrium. Of course it is described by the eigenvalues of linearization of the vector field on the orbit space. Alternatively we use the group action to view the relative equilibrium as an equilibrium in suitable coordinates, see Field [8]. The size of the eigenvalues gives the rate of contraction in directions transversal to the group orbit. If all these rates are nonzero then the group orbit is *normally hyperbolic*, see HIRSCH, PUGH & SHUB [15]. Normal hyperbolicity guarantees that the underlying manifolds persist as flow-invariant manifolds, see [15], Theorem 4.1. Generically near a bifurcation point these eigenvalues are proportional to certain powers of the distance to the bifurcation point. In order to ensure the persistence of the invariant manifold we allow only such perturbations which are sufficiently small to not destroy normal hyperbolicity, ie we require

$$|\varepsilon| < Cd^{\alpha}$$

for an appropriate choice of α. This will be our standing hypothesis. Observe that so far, the theory is the same for relative equilibria and for *relative periodic orbits.*

Definition 2. A relative periodic orbit is the preimage of a periodic orbit in the orbit space.

Observe that relative periodic orbits can be periodic themselves or they can be quasiperiodic.

If we want to describe briefly the effects of the perturbation on the dynamics on this object, then differences come up. Basically there are three cases. The most elementary setting is the description of the behavior on a perturbed relative equilibrium. Similar ideas apply to relative periodic orbits, which are periodic. The most difficult case is the one of an aperiodic relative periodic orbit.

First we want to describe briefly the situation for relative equilibria. Since, in Subsection 3.2 we do the same for periodic relative periodic orbits we just outline the procedure. We will discuss this last case in the main part of this paper.

Now we concentrate on the description of the most simple case: let G be a compact Lie group, $L = G_x$ be the isotropy subgroup of a point $x \in V$ and $H \subset G$ be the symmetry of the perturbed equation. Then we know that Gx is diffeomorphic to G/L and we consider H-equivariant vector fields on G/L. Important for the discussion are isotropy subgroups for this action and their fixed point subspaces. The details of these computations may be found in LAUTERBACH & ROBERTS [22]. We call a subgroup $K \subset H$ subconjugate to L, if there exists a $g \in G$ with $gKg^{-1} \subset L$. Then the isotropy subgroups for the action of H on G/L are just the subgroups which are subconjugate to L and the fixed point spaces are parameterized by the set of elements $N(K, L) = \left\{ g \in G \;\middle|\; gKg^{-1} \subset L \right\}$. So we have a complete description of the geometric properties of this action, for specific examples one can compute the relevant data, see also [22]. The main observation in this context is the appearance of heteroclinic cycles for specific choices of G, L, K ([22]). A systematic study when such cycles occur was carried out in [21].

2 Periodic and quasi-periodic solutions in equivariant systems

We are interested in trajectories of (2) that are periodic orbits *up to the group action*, ie corresponding to non-trivial periodic orbits $\widetilde{\Sigma}$ for the smooth projection of the vector field $\tilde{f}$ of f in the orbit space V/G and we denote its period by T. Since $\widetilde{f}$ is smooth, $\widetilde{\Sigma}$ is diffeomorphic to S^1 and is contained in the stratum $\widetilde{(H)}$ of an isotropy subgroup H for the action of G. If $\pi : V \mapsto V/G$ is the canonical projection on the orbit space, we denote by Σ the preimage $\pi^{-1}(\widetilde{\Sigma})$ and by Σ_H the intersection $\Sigma \cap V^{\mathrm{H}}$. Σ is called a *relative periodic orbit* for (2) and T its *relative period*. The group $\mathrm{N} = \mathrm{N}(H)/H$ acts freely on Σ_H and $\pi\Sigma_H = \widetilde{\Sigma}$. Let x_0 be a point in Σ_H and we denote its time orbit by $x(t)$ with $x(0) = x_0$. There is a point $x \in \Sigma_H$ and an element n in N such that $x = x(T) = nx_0$. The element n generates a monogenic subgroup $\widetilde{K}$ of N. It can be shown ([9,19]) that

$$\tilde{K}x_0 = \overline{\{x(\mathbb{R}) \cup Gx_0\}}$$

and furthermore that generically a connected component of Σ_H is a torus of dimension rank $\mathrm{N}(H)/H + 1$. In fact, there exist an element $k_0 \in \mathrm{N}(H)$ such that $x(T) = k_0x_0$ and it generates a monogenic subgroup group $K \subset \mathrm{N}(H)$ that projects down on $\widetilde{K}$ under the canonical projection $\mathrm{N}(H) \to \mathrm{N}(H)/H$; $K \subset \mathrm{N}(H)$ is a cyclic extension of H in $\mathrm{N}(H)$ and $K/H = \widetilde{K}$.

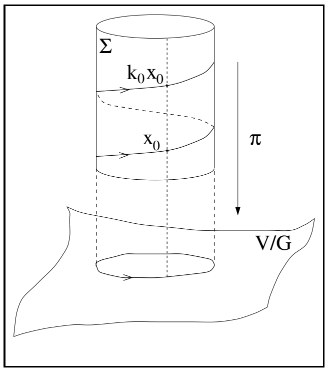

Fig. 2. A schematic representation of the structure of the (relative) periodic orbit Σ

Geometrically, H is the symmetry group of the point x_0 in Σ. Hence, all points in Σ have $[H]$ as isotropy type. The group generated by k_0 (whose closure is K) fixes the integral line of (2) passing through x_0 as a whole. Remark that if x_1 is a point in the group orbit of x_0 with $x_1 = gx_0$ then the group generated with gk_0g^{-1} fixes the integral line of (2) passing through x_1 as a whole. The conjugacy type of K corresponds to the spatio-temporal part of the symmetry. The relative periodic orbit is usually called a *standing wave* if $K = \{\mathbb{1}\}$ and a *discrete rotating wave* when $K = \mathbb{Z}_m$ for a given integer m.

3 Periodic orbits and their forced symmetry breaking

The simplest case of forced symmetry breaking for relative periodic orbits occurs when some trajectories in V are periodic orbits. However, the methods used to study forced symmetry breaking are geometric in nature since one can only get informations about the geometric structures that are forced to persist under the perturbation. In this respect, it turns out that forced symmetry breaking for relative periodic orbits that are not periodic orbits is very similar. We recall in this section the results obtained in [13] concerning the periodic case.

3.1 The geometry of periodic orbits

We assume now that Σ is a relative periodic orbit associated to a periodic orbit P through a point x_0 with relative period T. Of course, $x(\mathbb{R})$ project down onto $\widetilde{\Sigma}$ and $\Sigma = \{gx(t) | g \in G, t \in \mathbb{R}\}$. By periodicity, there is an integer n such that the period of P is $T_P = nT$. This is also the period of all trajectories in Σ. Rescaling the time in (2), we assume that $T_P = 2\pi$. This allow us to define a transitive action of $G \times \mathrm{S}^1$ on Σ as follows

$$(g, \alpha) \in G \times \mathrm{S}^1, y \in \Sigma, (g, \alpha) \circ y = g \cdot y(\alpha)$$

Since K acts on P, we can define a group homomorphism $\theta : K \mapsto \mathrm{S}^1$ such that

$$kx_0 = x(\theta(k))$$

with $\ker\theta = H$ and $\operatorname{im}\theta = \mathbb{Z}_m$. The isotropy subgroup Γ_{x_0} of the distinguished point x_0 for this action is then

$$\Gamma_{x_0} = \{(k, \theta(k^{-1})) | k \in K_0\}$$

and the relative periodic orbit Σ is diffeomorphic to $(G \times \mathrm{S}^1)/\Gamma_{x_0}$. On the other hand, one can consider the free right action of K on G making G a right principal K-bundle and the left K-action on S^1 given by

$$k \cdot \alpha = \alpha + \theta(k)$$

Now the quotient manifold of $G \times_K S^1$ under the diagonal action of K is the twisted product $\mathrm{E} = G \times_K \mathrm{S}^1$.

Any element in E can be uniquely represented as

$$[g, \alpha],\ \ g \in G, \alpha \in \mathrm{S}^1$$

and the projection $p : \mathrm{E} \mapsto G/K$ defines with $p([g, \alpha]) = [g]_K$ is a bundle projection for a bundle with total space E, base G/K, fiber S^1 and structure group K associated to the principal bundle $G \mapsto G/K$.

There is a G-equivariant diffeomorphism μ_{x_0} between E and Σ provided by (see [13], Propositions 3.7 and 3.8)

$$mu_{x_0}([g, \alpha]) = gx(\alpha)$$

Remark that with this parameterization, the time orbit of a point $y \in \Sigma$ project down on a unique point through the projection onto G/K. Indeed, any point y is uniquely written as $y = [g, \alpha]$ for $g \in G$ and $\alpha \in \mathrm{S}^1$. Now any real number $t \in \mathrm{I\!R}$ can be written as $t = qT + \beta$ with $q \in \mathbb{Z}$ and $\beta \in [0, 2\pi[$. Then $y(t) = k_0^q y(\beta)$ is represented as $[gk_0^q, \beta]$ and projects down on $[g]_K$ (k_0 is a generator of K). The flow induced from the flow on Σ by the diffeomorphism μ_{x_0} is then a vertical flow in E ([13] Theorem 4.19) ie it preserves the fibres of the bundle. More precisely, each fibre of the bundle is a single trajectory of the flow. Since E is locally diffeomorphic to $G/K \times \mathrm{S}^1$ (over an open set $\mathcal{U} \in G/K$), the flow associated to (2) is locally represented as

$$\left.\begin{array}{ll} \dot{[g]}_{\mathrm{K}} & = 0 \\ \dot{\alpha} & = r([g], \alpha) \end{array}\right\} \tag{3}$$

where r is a smooth vectorfield without fixed point on S^1.

Finally, one can show that, up to a G-equivariant diffeomorphism, the construction given above does not depend on the distinguished point x_0 on Σ.

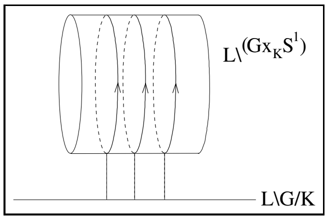

Fig. 3. A local representation of the geometry of the bundle $\mathrm{L}\backslash\mathrm{G} \times_{\mathrm{K}} \mathrm{S}^1 \mapsto \mathrm{L}\backslash\mathrm{G/K}$ with the induced periodic flow for the unperturbed system.

3.2 Forced symmetry breaking for periodic orbits

The information we can obtain concerning forced symmetry breaking using the methods developed in [22] are based on the characterization of the geometric structure that persist under the perturbation. We now assume that the relative periodic orbit Σ in (2) is normally hyperbolic and one considers the new system

$$\dot{x} = f(x) + \epsilon h(x) \tag{4}$$

with ϵ small and h a C^r L-equivariant vector field with $r \geq 1$. Then Proposition 1.1 in [22] ensures that if ϵ is small enough, there exists a unique C^r-manifold Σ_ϵ near Σ which is invariant under the flow associated to (4) and which is L-equivariantly C^r-diffeomorphic to Σ. Classifying the geometric structure corresponding to fixed point subspaces for the isotropy subgroups of L on Σ_ϵ can allow to obtain the necessary existence of fixed points for the flows, or possibility of heteroclinic connections or heteroclinic cycles etc$\cdots$ (see [22,20,21,24]). Since Σ_ϵ and Σ are L-equivariantly diffeomorphic, we can undertake this characterizations on Σ itself for which we have an explicit algebraic representation.

Since G acts on a point $g_0 x_0(t_0)$ as $g \circ g_0 x_0(t_0) = g g_0 x_0(t_0)$, the G action induced on E is given by

$$g \circ [g_0, t_0] = [g g_0, t_0]$$

The subgroup $L \subset G$ of symmetry of the perturbation acts on E by restriction of the action of G. Let H_1 and H_2 two subgroups of G and let $\mathrm{N}(H_1, H_2)$ be defined by

$$\mathrm{N}(H_1, H_2) = \{g \in G \mid H_1 \subset g H_2 g^{-1}\}.$$

If $\mathrm{N}(H_1, H_2) \neq \emptyset$ then H_1 is said to be *subconjugated* to H_2. Then fixed points subspaces for the L-action on E are characterized by (see [13] Proposition 4.13):

Proposition 3. *Let L' be a subgroup of L. We have*

(a) $\mathrm{Fix}(L') \neq \emptyset \Leftrightarrow \mathrm{N}(L', H) \neq \emptyset$
(b) $\mathrm{Fix}(L') = [\mathrm{N}(L', H), \mathrm{S}^1]$

In particular, part (*b*) of the proposition shows that the fixed point subspaces are vertical spaces relatively to the bundle structure of E and all points in a periodic orbit on Σ will still be in the same strata for the action of L after the perturbation.

Since L acts on the left E, on can perform a orbit space projection on the orbit space $L\backslash\mathrm{E}$. The fact that the left action of L on E and indeed on G commutes with the right K-action on G inside the twisted product $G \times_K \mathrm{S}^1$ allows to show the following ([13] Theorem 4.17) :

T *he mapping $\pi_0 : L\backslash(G \times_K \mathrm{S}^1) \mapsto L\backslash G/K$ defines a fiber bundle with fiber S^1 and structure group K.*

Furthermore, using the form (3) of the unperturbed vector field (ie $\epsilon = 0$), one can show the following ([13] Theorem 4.19 and Proposition 4.20)

T *he projection on $L\backslash\mathrm{E}$ of the vector field on* E *induced by $f|_{\Sigma}$ is a vertical vector field relatively to the bundle structure on $L\backslash\mathrm{E}$. Furthermore, all solutions of $\widetilde{f}$ are T/n periodic with n the order of the cyclic group $\theta(L\cap K)$.*

Remark that since the fibers in $L\backslash\mathrm{E}$ are isomorphic to S^1, the last result implies that each fiber in $L\backslash\mathrm{E}$ is a periodic orbit and then there is a one-to-one correspondence between the points in the base $L\backslash G/K$ and the periodic orbits of the unperturbed vector field. The restriction of the initial flow to $L\backslash\mathrm{E}$ can be thought of as a global cross-section of the set of periodic orbits in Σ (remark however that such global cross-section does not necessarily exists as a bundle-section since $\mathrm{E} \mapsto L\backslash\mathrm{E}$ has no reason to be a trivial bundle). In conclusion, for the perturbed system (4) with ϵ small enough one has :

- 0-dimensional strata in $L\backslash G/K$ correspond to periodic orbits persisting to the perturbation
- 1-dimensional strata with boundaries indicate the possibility of homoclinic or heteroclinic trajectories
- 1-dimensional strata, compact and without boundary, indicate the possibility of relative periodic orbits

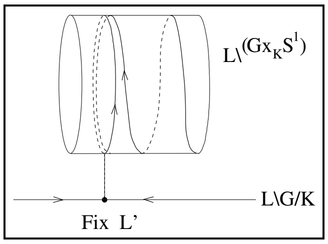

Fig. 4. A local representation of a flow for the perturbed system. Here there is a persisting periodic orbit associated with the point $\mathrm{Fix}(L')$ in $L\backslash G/K$. A typical trajectory outside this fixed point is also sketched.

4 Forced symmetry breaking for relative periodic orbits

In this section, we show that the problem of forced symmetry breaking for relative periodic orbits can be brought back to the same problem for periodic orbits.

4.1 The geometry of relative periodic orbits

In order to describe the geometric structure of relative periodic orbits, we go back to their definition : a relative periodic Σ of isotropy type H is the preimage by the orbit map $\pi : V \mapsto V/G$ of a smooth non-trivial periodic orbit $\widetilde{\Sigma}$ in the strata $\widetilde{(H)}$ of H. When no confusion is possible, we denote with π the canonical orbit map as well as it restriction on Σ. Since the action of G is smooth, one has the following result ([12]) :

T *he mapping* $\pi\ : \Sigma \mapsto \widetilde{\Sigma}$ *defines a fiber bundle with base* S^1, *fiber* G/H *and group* $\mathrm{N} = \mathrm{N}(H)/H$ *associated to the principal* N*-bundle* $\pi\ : \Sigma_H \mapsto \widetilde{\Sigma}$, *ie*

$$\Sigma = (G/H) \times_{\mathrm{N}} \Sigma_H$$

The classification of possible relative periodic orbit with isotropy type H is then equivalent to the classification of principal N-bundles over S^1. According to the general scheme of classification for bundles over spheres ([29]), on can show the following ([14]) :

Proposition 4. *The set* $\mathrm{P}(\mathrm{N}, \mathrm{S}^1)$ *of isomorphism of principal* N*-bundles over* S^1 *is in one-to-one correspondence with the set of conjugacy classes in* N/N_0 *with* N_0 *the connected component of the identity in* N.

If x_0 is a distinguished point in Σ such that $x(T) = n_0 x_0$ with $n_0 \in \mathrm{N}$, then n_0 project down to an element $\widetilde{n}_0$ in N/N_0 whose conjugacy class characterizes the isomorphism type of the corresponding relative periodic orbit Σ.

In fact, more informations can be found concerning the geometry if one notice that the vector field on Σ is smooth and non-singular and thus provides a smooth 1-foliation on Σ. Therefore, the fiber bundle $\Sigma \mapsto \widetilde{\Sigma}$ is an example of foliated bundle and is *flat*, meaning that its structure groups can be reduced to a discrete one. If $\widetilde{n}_0$ is the element of N/N_0 defined above, then its generates a cyclic subgroup $\widetilde{\Gamma}(\widetilde{n}_0) \subset \mathrm{N}/\mathrm{N}_0$ of a given order q. It can be shown (Lemma 4.2 [9]), that there is a cyclic subgroup Γ of order q in N projecting down on $\widetilde{\Gamma}$ in N/N_0. If γ is a generator of Γ then one has $x(T) = n_0 x_0 = \gamma n_0' x_0$ with $n_0' \in \mathrm{N}_0$. One has the following result ([14]) :

Lemma 5. *The structure group of the bundle* $\pi : \Sigma_H \mapsto \widetilde{\Sigma}$ *can be reduced to* $\Gamma(n_0)$ *and* Σ_H *can be written as*

$$\Sigma_H = \mathrm{N} \times_\Gamma \mathrm{S}^1$$

and is the total space of a smooth fiber bundle with fiber N *associated to the smooth principal* Γ*-bundle* $p : \mathrm{S}^1 \mapsto \mathrm{S}^1$ *corresponding to the* q*-covering of* S^1.

Using [9] Lemma 4.2 (replacing G_0 by H), there is a subgroup $\Gamma' \subset \mathrm{N}(H)$ isomorphic to Γ and covering it for the mapping $\mathrm{N}(H) \mapsto \mathrm{N}(H)/H$. We denote with K the product $\Gamma' H$. Since Γ' acts on the right on G/H, one can define an action of Γ on G/H with

$$[g]_H\gamma = [g\gamma']_H$$

where γ' projects down on γ. Then we can identify Γ and Γ' when there are no possible confusions. As a consequence, the relative periodic orbit Σ can be seen as the total space of the Γ-principal bundle $(G/H) \times_\Gamma \mathrm{S}^1$. The elements of Σ can be uniquely represented as

$$[[g]_H, \alpha],\ g \in G,\ \alpha \in S^1$$

with the additional condition

$$[[g]_H, \alpha + 2\pi/q] = [[g\gamma]_H, \alpha]$$

We can use the flow associated to (2) to define a connection on Σ, ie an smooth and equivariant distribution of "horizontal" tangent spaces transversal to the fibers. Here this connection is given by tangent vectors to the flow. Let $\omega : [0, 2\pi] \mapsto \widetilde{\Sigma}$ be a smooth loop in $\widetilde{\Sigma}$ with base point $\pi(x_0)$ in $\widetilde{\Sigma}$. It can be lifted in a unique integral flow line s_ω of (2) on Σ with $s_\omega(0) = x_0$. Let $\Omega\widetilde{\Sigma}$ be the space of smooth loops in $\widetilde{\Sigma}$. Then, the mapping $h : \Omega\widetilde{\Sigma} \mapsto \mathrm{N}(H)/H$ define with $\omega \longrightarrow g$ such that $s_\omega(1) = gx_0$ is called the *holonomy* of the bundle. It clearly depends on the flow associated to (2). However, the conjugacy class (under inner automorphism of $\mathrm{N}(H)/H$) of the homotopy class of h denoted $h(\Sigma)$ and called the *holonomy map* is independent of this flow and is a geometric invariant of the group action (see [16] Theorem 1.1). With the previous construction, this holonomy map assigns γ to the generator of $\Omega\widetilde{\Sigma} = \Omega\mathrm{S}^1$. We call Γ the *holonomy group* of Σ. The (conjugacy class of the) holonomy group completely characterizes the type of the bundle corresponding to Σ (compare with Proposition 4). Using properties of fiber bundles, on can show the following ([12]) :

1. *If $\widetilde{\Sigma} \subset V/G$ is contractible in the strata of H, then the holonomy map $h(\Sigma)$ is trivial and $\Sigma \simeq G/H \times \mathrm{S}^1$.*
2. *If Σ and Σ' are two relative periodic orbits with isotropy H for the system (2) such that there respective projection $\widetilde{\Sigma}$ and $\widetilde{\Sigma}'$ are homotopically equivalent within $(\widetilde{H})$, then Σ and Σ' have conjugated holonomy group.*

4.2 Forced symmetry breaking for relative periodic orbits

Since the principal Γ-bundle $\mathrm{S}^1 \mapsto \mathrm{S}^1$ defined in Lemma 5 is smooth, one can find a smooth $2\pi q$-periodic flow covering the non trivial 2π periodic flow on S^1. It induces a G-equivariant $2\pi q$-periodic flow on Σ covering the 2π-periodic flow on $\widetilde{\Sigma}$:

$$\forall\, t \in \mathrm{I\!R}, \forall\, [[g]_H, \alpha] \in (G/H) \times_\Gamma \mathrm{S}^1,\ t \circ [[g]_H, \alpha] = [[g]_H, \alpha + t \bmod 2\pi/q]$$

(recall that $\gamma^q = \mathrm{e}$). Here $[[\mathrm{e}]_H, 0]$ is identified with the distinguish point x_0. Details of this construction are given in [14]. As a conclusion, one can always find a periodic flow on a relative periodic orbit. Now the subgroup $\Gamma \subset \mathrm{N}(H)$ is precisely the subgroup of G fixing the periodic orbit through $[[\mathrm{e}]_H, 0]$ as a set. Indeed, this periodic orbit is given with $\{[[\mathrm{e}]_H, \alpha],\ \alpha \in [0, 2\pi q]\}$ and if γ' generates Γ' and $h \in H$, one has

$$\gamma' h[[\mathrm{e}]_H, \alpha]] = [[\gamma']_H, \alpha]] = [[\mathrm{e}]_H \gamma, \alpha]] = [[\mathrm{e}]_H, \alpha + 2\pi]$$

One then obtain the following :

The problem of forced symmetry breaking for relative periodic orbits with holonomy group Γ is "equivalent" to the problem of forced symmetry breaking for periodic orbits with spatio-temporal symmetry $K = \Gamma H$ (or $\widetilde{K} = \Gamma$). More precisely we have:

From the point of view of the persistence of geometric structures, the problem of forced symmetry breaking for relative periodic orbits with holonomy group Γ can be brought back to the problem of forced symmetry breaking for periodic orbits with spatio-temporal symmetry $K = \Gamma H$ (or $\widetilde{K} = \Gamma$).

Another way to see this point is that the total space $(G/H) \times_\Gamma \mathrm{S}^1$ of the bundle associated to Σ is G-equivariantly and smoothly isomorphic to $G \times_{\Gamma H} \mathrm{S}^1$ through the mapping

$$[[g]_H, \alpha] \longrightarrow [g, \alpha]$$

The proof that this mapping is well defined is straightforward and only uses the definitions and the fact the Γ and H commute.

5 An application

In this section we want to discuss some related results for special dynamical systems without going into any detail. The details can be found in the literature. Observe that the dynamics close to a heteroclinic cycle is characterized by sudden bursts being followed by long periods of rest. A typical example for this type of motion is the change of polarity of the earth's magnetic field. Measurements (of magnetism in solidified lava) indicate that the orientation of the field is constant on time scales of 10^4 to 10^6 years and that a reversal takes several hundred years. We do not want to go into details of this fascinating subject (see for example Childress [2], Childress & Gilbert [3] or Ghil & Childress [11] for further background and more references). The problem for the underlying mechanism is still unsolved. Here, we want to describe some recent research on the occurrence of heteroclinic cycles in magnetohydrodynamics. The underlying equations are the Maxwell equations coupled with the Navier-Stokes equations. A first (very crude) approximation of the problem might be to look at these equations in a spherical shell and to neglect the force on the fluid motion due to the magnetic field. If we do this we might consider the Navier-Stokes equations in a shell and take solutions of these equations to compute the resulting magnetic field (Observe that there are some problems due to the fact that this is a linear equation.) from the induction equation.

Performing a center manifold reduction one reduces the problem to a low-dimensional ordinary differential equation. Restricting to the the spherical case this kind of reduction has been done at various places, see Chossat [4,5]. In [7] it has been shown that near the onset of convection no complicated dynamics can develop if the representation of the group $\mathbf{O}(3)$, the symmetry group of the sphere, acts absolutely irreducibly on the critical modes. Therefore one might ask, whether there are parameter values where we see mode interactions generically. This question has been answered positively by Armbruster & Chossat [1]. They also proved the existence of a heteroclinic cycle in the nonrotating case for the eight-dimensional representation of $\mathbf{O}(3)$ which is a sum of the three and five-dimensional irreducible representation. This cycle involves two types of axisymmetric solutions: one type is characterized by an inflow along the equator and an outflow at the poles and the other way around at the other type. This burst-like dynamics was already predicted by a numerical study of Friedrich & Haken [10]. However, the solutions found numerically do not completely agree with the analysis in [1]. There are qualitative differences, see [6]. In [6] it was also shown that the burst-like behavior found by Friedrich & Haken [10] can be explained by the existence of a large flow-invariant set involving heteroclinic cycles. This set is crucial for the stability analysis of the heteroclinic cycles found by Armbruster & Chossat [1]. All this applies to the nonrotating sphere. Rotation can be modelled as a forced symmetry breaking. This implies that we are looking at very slow angular velocities. Then it is shown that we still find heteroclinic behavior, however it involves not only equilibria but also rotating waves. Of course the rotation in the case of the earth is not slow, so these results are not applicable to the question of pole reversals.

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