Cascades of homoclinic doubling bifurcations

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Abstract. We present an overview of the theory of homoclinic doubling cascades, describing bifurcation theory and discussing universal scaling properties obtained from a renormalization theory.

1 Introduction

Cascades of homoclinic doubling cascades can occur in two parameter families of three dimensional vector fields and lead to the creation of homoclinic orbits with arbitrarily long arclength. Particular classes of vector fields where we think that homoclinic doubling cascades possibly occur are given by 'hooked Lorenz models', studied recently in [LuzVia99,LuzTuc99], and Shimizu-Morioka models [Shi93]. The models discussed in this review, where homoclinic doubling cascades have been shown to occur, are similar to those models.

Recall that the Lorenz model provides an example of a strange attractor for a set of differential equations that contains an equilibrium. Due to the partially hyperbolic structure of this attractor (there exists a continuous splitting of the tangent bundle along the attractor in strong stable and center unstable directions), a good description of the dynamics is possible. In contrast, the strange attractors appearing in the Hénon map where no such partial hyperbolic structure exists, are much harder to grasp. In the Lorenz model, a first return map on a cross section maps this cross section to two wedge shaped regions. Dynamics of a more complicated nature than in the Lorenz model, which still involves the equilibrium, occurs when these wedges are bended when they return to the cross section. A partially hyperbolic structure no longer exists for such hooked Lorenz maps. In fact, the resulting model is something of a combination of the Hénon model and the Lorenz model.

Where the Lorenz model possesses a (approximate) symmetry, so that all points of the cross section return, we will consider models as depicted in figure 1 where only one half of the cross section returns; we assume it returns as a hook as indicated in the figure. A fascinating phenomenon can occur in such situations when allowing to vary the differential equations with parameters: the occurrence of cascades of homoclinic doubling equations. This phenomenon is outlined in this paper. Of course, when studying symmetric vector fields the same phenomenon will also be present.

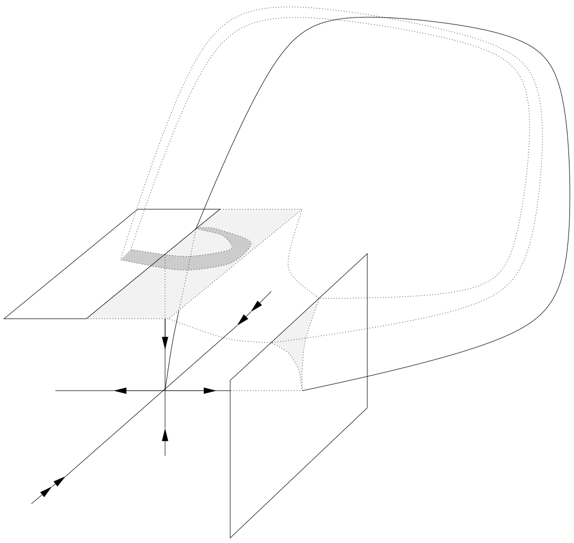


Fig. 1. We consider unfoldings of vector fields with a homoclinic orbit for which the first return map on a cross section maps a strip to a hook.

The vector fields we will consider are obtained from a bifurcation study, in which a vector field with a degenerate homoclinic orbit of codimension three is unfolded. This approach has a number of advantages. Because all the dynamics occurs near the degenerate homoclinic orbit, the depicted hook is small and near the stable manifold of the equilibrium. Since moreover in our bifurcation study the divergence of the vector fields at the equilibrium is negative, the first return map on the cross section is strongly area contracting. As a consequence the dynamics is close to one dimensional, and much can be understood from the study of a derived interval map. All of this will be explained in section 3. In the following section we introduce the homoclinic doubling bifurcation. Section 4 discusses a pathfollowing theory for homoclinic orbits; this theory is an important ingredient in existence proofs for homoclinic doubling cascades. Finally, in section 5 we study bifurcation diagrams and in particular scaling structures in the bifurcation set. This is done using renormalization theory.

A numerical study of vector fields as considered in this paper has recently been presented in [OldKraCha00]. This numerical study confirms the predictions of the theory, including the universal scaling structures.

2 Homoclinic doubling cascades

We will first state a precise bifurcation theorem for the homoclinic doubling bifurcation. After that we start discussing cascades of homoclinic doubling bifurcations.

2.1 Homoclinic doubling bifurcation

Let X be a smooth vector field on $\mathbb{R}^3$ with a hyperbolic equilibrium q. We assume that $DX(q)$ has two distinct real stable eigenvalues and one unstable eigenvalue. By a time reparametrization, we may assume the unstable eigenvalue to be equal to 1. Write $-\alpha, -\beta$ with $\alpha > \beta > 0$ for the two stable eigenvalues. Because the two stable eigenvalues are distinct, the vector field X has, contained in the stable manifold $W^{ss,s}(q)$ of q, a one dimensional strong stable manifold $W^{ss}(q)$. Its tangent space at q is the eigenspace associated to $-\alpha$. There further exists a two dimensional center unstable manifold $W^{s,u}(q)$ with tangent space at q spanned by the eigenspaces associated to the eigenvalues $-\beta$ and 1. This last invariant manifold is not unique and in general it is only C^1 [HirPugShu77]. The tangent bundle of (any) $W^{s,u}(q)$ along the unstable manifold, however, is a uniquely determined smooth bundle, see e.g. [Hom96] for a proof of this fact. All these invariant manifolds persist under perturbation of the vector field. We write $W_X^{ss,s}(q), W_X^{s,u}(q)$, etc., if we wish to stress the dependence of these manifolds on the vector field X.

Definition 1. Suppose Γ is a homoclinic orbit of X, that is, a nontrivial intersection of $W^{ss,s}(q)$ with $W^u(q)$. Γ is called an *inclination-flip* homoclinic orbit, if $W^{ss,s}(q)$ is tangent to one (and hence any) center unstable manifold $W^{s,u}(q)$ along Γ.

Inclination-flips are examples of homoclinic bifurcations of (at least) codimension two. We will not precise the notion of codimension. What is heuristically meant with a bifurcation being of codimension n, is that it is given by a collection of conditions, naturally occurring in its study, that make up a manifold of codimension n in the space of vector fields.

Consider a smooth two parameter family of vector fields $\{X_\gamma\}, \gamma \in \mathbb{R}^2$, on $\mathbb{R}^3$, satisfying the following conditions:

(BH: Basic hypothesis) The vector field X_γ has a hyperbolic equilibrium q_γ at which the linearization $DX_\gamma(q_\gamma)$ possesses two negative eigenvalues $-\alpha(\gamma) < -\beta(\gamma)$ and one positive eigenvalue 1.

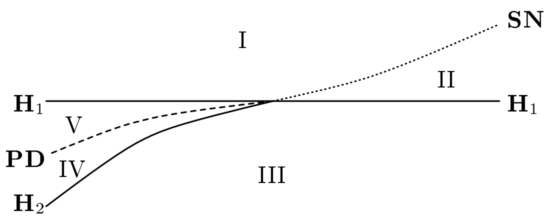
(IF: Inclination-flip) The vector field X_0 possesses, at the parameter $\gamma = 0$, an inclination-flip homoclinic orbit Γ. The homoclinic orbit Γ is not contained in the strong stable manifold $W_{X_0}^{ss}(q_0)$.

(EC: Eigenvalue conditions) The eigenvalues of the linearization $DX_0(q_0)$ satisfy

$$\alpha(0) > 1 \text{ and } \frac{1}{2} < \beta(0) < 1.$$

(GU: Generic unfolding) Denote by $F_\gamma^{s,u}$ the bundle $\{T_x W_{X_\gamma}^{s,u}(q_\gamma);\ \ x \in W_{X_\gamma}^u(q_\gamma)\}$. The condition is then that $\bigcup_\gamma (TW_{X_\gamma}^{ss,s}(q_\gamma),\gamma)$ and $\bigcup_\gamma (F_\gamma^{s,u},\gamma)$ intersect each other transversally along $T_\Gamma W_{X_0}^{ss,s}(q_0) \times \{0\}$ in $T\mathbb{R}^3 \times \mathbb{R}^2$.

Theorem 2. [**KisKokOka93,HomKokNau98**] *Let $\{X_\gamma\}$ be a two parameter family of vector fields on $\mathbb{R}^3$ as above. After a reparametrization of the parameter plane, the bifurcation diagram of $\{X_\gamma\}$ for small values of γ, is as depicted below. ¿From the curve $\mathbf{H}_1$ of primary homoclinic orbits, a curve*



$\mathbf{H}_2$ of doubled homoclinic orbits branches. Furthermore, a curve **SN** *of periodic saddle-node bifurcations and a curve* **PD** *of period-doubling bifurcations branch.*

The following list describes all periodic orbits of $\{X_\gamma\}$ in a tubular neighborhood of $\overline{\Gamma}$, for parameters from the different regions I,...,V. In region I, $\{X_\gamma\}$ has no periodic orbits. In region II, $\{X_\gamma\}$ has an attracting 1-periodic orbit and a saddle 1-periodic orbit. In region III, $\{X_\gamma\}$ has an attracting 1-periodic orbit. In region IV, $\{X_\gamma\}$ has an attracting 1-periodic orbit and a saddle 2-periodic orbit. In region V, $\{X_\gamma\}$ has a saddle 1-periodic orbit and an attracting 2-periodic orbit.

2.2 Cascades of homoclinic doubling bifurcations

When each doubled homoclinic orbit that is created in a homoclinic doubling bifurcation undergoes itself, at some parameter value, a homoclinic doubling bifurcation, then a cascade of homoclinic doubling bifurcations is created in which homoclinic orbits of arbitrarily long arclength occur. Though precise statements will be made in the next section, we state here informally that homoclinic doubling cascades occur persistently in two parameter families of vector fields.

Theorem 3. [**HomKokNau98**] *In the space of two parameter families of smooth vector fields on $\mathbb{R}^3$ there is an open set consisting of families that possess a cascade of homoclinic-doubling bifurcations.*

The following sections are devoted to a description of families of vector fields in which homoclinic doubling cascades occur persistently, plus a discussion of the resulting bifurcation set.

3 A bifurcation theorem; the resonant inclination flip

Consider a smooth three parameter family of vector fields $\{X_\gamma\}, \gamma \in \mathbb{R}^3$, on $\mathbb{R}^3$ satisfying the following conditions:

(BH: Basic hypothesis) The vector field X_γ has a hyperbolic equilibrium q_γ at which the linearization $DX_\gamma(q_\gamma)$ possesses two negative eigenvalues $-\alpha(\gamma) < -\beta(\gamma)$ and one positive eigenvalue 1.

(IF: Inclination-flip) The vector field X_0 possesses, at the parameter $\gamma = 0$, an inclination-flip homoclinic orbit Γ. Γ is not contained in $W^{ss}_{X_0}(q)$.

(SR: Strong resonance) The eigenvalues of the linearization $DX_0(q_0)$ satisfy

$$\beta(0) = \frac{1}{2} \text{ and } \alpha(0) > 1.$$

(GU: Generic unfolding) $\bigcup_\gamma (F^{s,u}_\gamma, \beta(\gamma), \gamma)$ and $\bigcup_\gamma (TW^{ss,s}_{X_\gamma}(q_\gamma), \beta(\gamma), \gamma)$ intersect transversally along $T_\Gamma W^{ss,s}_{X_0}(q_0) \times \{1/2\} \times \{0\}$ in $T\mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}^3$.

Take coordinates (x_{ss}, x_s, x_u) near q_γ in which

$$DX_\gamma(q_\gamma) = -\alpha x_{ss}\frac{\partial}{\partial x_{ss}} - \beta x_s \frac{\partial}{\partial x_s} + x_u \frac{\partial}{\partial x_u}.$$

Let $\Sigma^{in} \subset \{x_s = \delta\}$ with δ small and positive, be a cross-section transverse to Γ. By a linear rescaling, we may assume $\delta = 1$.

Proposition 5 below gives expansions for the Poincaré return map. These expansions hold in suitable smooth coordinates provided by the following normal form result. Let X_γ be given by a set of ordinary differential equations

$$\begin{aligned} \dot{x}_{ss} &= -\alpha x_{ss} + F_{ss}(x_{ss}, x_s, x_u; \gamma), \\ \dot{x}_s &= -\beta x_s + F_s(x_{ss}, x_s, x_u; \gamma), \\ \dot{x}_u &= x_u + F_u(x_{ss}, x_s, x_u; \gamma), \end{aligned} \tag{3.1}$$

where F^{ss}, F^s, F^u are quadratic and higher order terms.

Lemma 4. [**HomKra98**] *The vector field X_γ is smoothly equivalent to a vector field of the same form with*

$$\begin{aligned} F_{ss}(x_{ss}, x_s, x_u; \gamma) &= \mathcal{O}(\|(x_{ss}, x_s)\|^2), \\ F_s(x_{ss}, x_s, x_u; \gamma) &= x_u \mathcal{O}(|x_{ss}| + \|(x_{ss}, x_s)\|^2), \\ F_u(x_{ss}, x_s, x_u; \gamma) &= 0. \end{aligned}$$

Proposition 5. [**HomKra98**] *In coordinates given by lemma 4, $\Phi : \Sigma^{in} \to \Sigma^{in}$ has the following expression:*

$$\Phi(x_{ss}, x_u; \gamma) = \begin{pmatrix} Q(\gamma) + A(\gamma)x_u^\beta + B(x_{ss};\gamma)x_u^{2\beta} + R^{ss}(x_{ss}, x_u; \gamma) \\ \mu_2(\gamma) + \mu_1(\gamma)x_u^\beta + D(x_{ss};\gamma)x_u^{2\beta} + R^u(x_{ss}, x_u; \gamma) \end{pmatrix} \tag{3.2}$$

Here μ_1, μ_2, Q, A are smooth functions of γ, with μ_1, μ_2 vanishing at $\gamma = \gamma_0$. The functions B, D are smooth in x_{ss} and γ. Furthermore, R^{ss} and R^u are smooth for $x_u > 0$; for some $\sigma > 0$, there exist constants $C_{k+l} > 0$ so that with $i = ss, s$

$$\left| \frac{\partial^{k+l}}{\partial x_u^k \partial (x_{ss}, \gamma)^l} R^i(x_{ss}, x_u) \right| \leq C_{k+l} x_u^{2\beta+\sigma-k} .$$

Define

$$\mu_3(\gamma) = 2 - \frac{1}{\beta(\gamma)}.$$

The generic unfolding condition (GU) is equivalent to

$$\frac{\partial}{\partial \gamma}(\mu_1, \mu_2, \mu_3)(\gamma)|_{\gamma=\mathbf{0}} \neq 0.$$

One may therefore, by reparametrizing the parameter space, assume that $\gamma = (\mu_1, \mu_2, \mu_3)$. Note that a 1-homoclinic orbit exists for $\mu_2 = 0$, which is an inclination-flip homoclinic orbit if also $\mu_1 = 0$.

Theorem 6. [**HomKra98**] *Let $\{X_\gamma\}$, $\gamma = (\mu_1, \mu_2, \mu_3) \in \mathbb{R}^3$, be a smooth three parameter family of vector fields as above. With D as in (3.2), suppose that $D > 1$ at $\gamma = 0$. For each μ_1 sufficiently small and negative, the two parameter family $\{Y_{\mu_2,\mu_3}\}$ given by $Y_{\mu_2,\mu_3} = X_{\mu_1,\mu_2,\mu_3}$, possesses a connected set of homoclinic bifurcation values in the (μ_2, μ_3)-parameter plane, containing a cascade (μ_2^n, μ_3^n) of homoclinic-doubling bifurcations (inclination-flips) in which a 2^n-homoclinic orbit is created.*

Note that the value of D in (3.2), for $\gamma = (0, 0, 0)$, is independent of the choice of cross section Σ. Indeed, varying Σ corresponds to conjugating Φ with a diffeomorphism. This is easily seen to leave D invariant, if β equals $\frac{1}{2}$.

3.1 Singular rescalings to interval maps

A rescaling brings the Poincaré return map to a map which is a small perturbation of an interval map. Let Φ be as in (3.2) and define rescaled coordinates $(\hat{x}_{ss}, \hat{x}_u)$ by

$$\begin{aligned} x_{ss} - Q &= |\mu_1| \hat{x}_{ss}, \\ x_u &= \left| \frac{\mu_1}{2D} \right|^{1/\beta} \hat{x}_u. \end{aligned}$$

The following proposition, which gives expansions for the first return map in rescaled coordinates, is proved by a direct computation.

Proposition 7. **[HomKra98]** *Let $\hat{\Phi}$ be the Poincaré return map in the rescaled coordinates $(\hat{x}_{ss}, \hat{x}_u)$. Write $r = |\mu_1|^{\mu_3}/D$ and $p = r(4D\mu_2|\mu_1|^{-2}-1)$. Then*

$$\hat{\Phi}(\hat{x}_{ss}, \hat{x}_u) \to \begin{pmatrix} \frac{A}{2D}\hat{x}_u^{\beta} + R_1(\hat{x}_{ss}, \hat{x}_u) \\ p + r(1-\hat{x}_u^{\beta})^2 + R_2(\hat{x}_{ss}, \hat{x}_u) \end{pmatrix},$$

where R_1 and R_2 converge to 0 as $\gamma \to 0$.

As $\mu_1 \to 0$, restricting $\hat{x}_u$ to a compact interval and parameters (μ_1, μ_2, μ_3) from a chart near the origin on which $|p|, |r|$ are bounded, $\hat{\Phi}$ converges to

$$\hat{x}_u \mapsto \begin{pmatrix} \frac{A}{2D}\hat{x}_u^{\beta} \\ p + r(1-\hat{x}_u^{\beta})^2 \end{pmatrix}. \tag{3.3}$$

Ignoring the first coordinate leads to a unimodal map on an interval. Note that homoclinic orbits for the vector field correspond to periodic orbits through 0 for the interval map. Homoclinic doubling bifurcations correspond to periodic orbits that contain both 0 and the critical point.

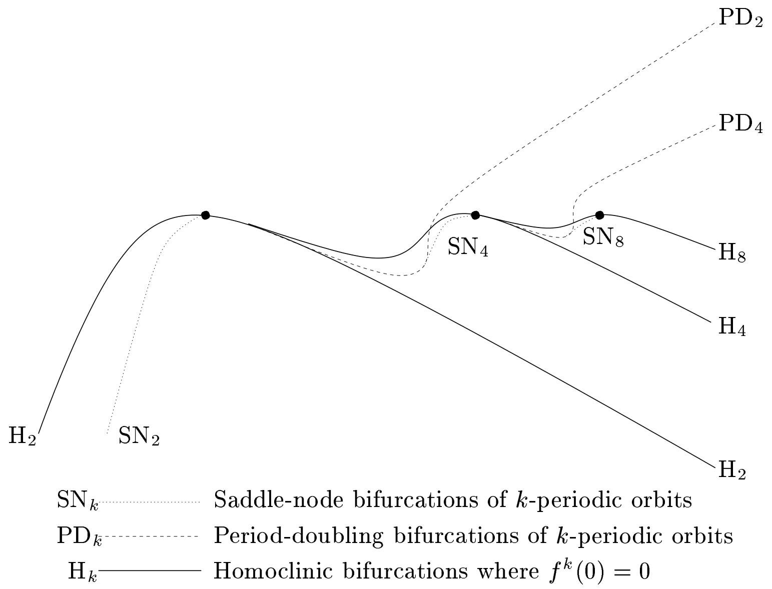


Fig. 2. Schematic impression of the expected bifurcation set of $x \mapsto p + r(1-x^{\beta})^2$ in the parameter plane $\{(r,p) \in \mathbb{R}^2\}$. The thick dots indicate three subsequent 'homoclinic doubling' bifurcations.

A study of the family of interval maps (3.3) suggests what to expect for the original vector fields. It is not hard, using kneading theory, to see that the family of interval maps possesses a cascade of homoclinic doubling bifurcations.

Indeed, consider the one parameter family of interval maps $\hat{x}_u \mapsto r(1-\hat{x}_u^\beta)^2$ obtained by letting $p=0$ in (3.3). Expressed in the coordinate $u=\hat{x}_u^\beta$ this leads to the C^1 family $u \mapsto r(1-u)^{2\beta}$. Such families are known to possess a cascade of parameter values with periodic orbits of period 2^n and containing the critical point. These parameter values correspond to homoclinic doubling bifurcations.

To show that the first return maps, i.e. the vector fields, possess a cascade of homoclinic doubling bifurcations one relies on a continuation theory for homoclinic orbits. This is discussed in the next section. There exists much similarity between proving the existence of period doubling cascades in smooth families of unimodal maps (using kneading theory) and in families of planar diffeomorphisms such as dissipative Hénon families [YorAll83] (relying on a continuation theory for periodic orbits).

4 Pathfollowing homoclinic orbits

In this section we discuss a general pathfollowing theory for homoclinic orbits. Below, in section 4.1, we indicate how this pathfollowing theory is used to prove the existence of homoclinic doubling cascades in theorem 6. An earlier approach to pathfollowing homoclinic orbits is found in [Fie96].

Let $\mathfrak{X}$ be the set of smooth vector fields on $\mathbb{R}^3$, equipped with the weak Whitney topology. Consider the set $\mathfrak{X}_2$ of smooth two parameter families of smooth vector fields on $\mathbb{R}^3$. Let $\{X_\lambda\}$ be a two parameter family of vector fields from $\mathfrak{X}_2$, depending on a parameter $\lambda \in \mathbb{R}^2$.

Let $\mathcal{P}$ be the set of bounded closed subsets of $\mathbb{R}^3$, equipped with the Hausdorff metric. Let

$$G = \left\{ (\mu, h) \in \mathbb{R}^2 \times \mathcal{P}; \begin{array}{l} h \text{ is the union of an equilibrium and a} \\ \text{homoclinic orbit of } X_\mu \end{array} \right\}. \tag{4.4}$$

For $(\mu,h) \in G$, let $l(\mu,h)$ denote the length of h. We say that k is a virtual length of (μ,h), if there exists a sequence of perturbations $\{Y_\lambda^i\}$ of $\{X_\lambda\}$ with $\{Y_\lambda^i\} \to \{X_\lambda\}$ as $i \to \infty$ so that $\{Y_\lambda^i\}$ possesses a homoclinic orbit h_i at parameter values μ_i with $\mu_i \to \mu$, $h_i \to h$ in the Hausdorff topology and $l(\mu_i,h_i) \to k$ as $i \to \infty$. We write $\tau(\mu,h)$ for the set of virtual lengths of (μ,h).

Let $(\mu,h) \in G$ so that h is the union of a homoclinic orbit and a hyperbolic equilibrium. Write Γ for the connected component of G containing (μ,h). We call (μ,h) globally continuable if either

- $\Gamma \backslash \{(\mu,h)\}$ is connected

or each component C of $\Gamma \backslash \{(\mu,h)\}$ satisfies at least one of the following conditions:

- C is unbounded

– there exists a sequence $(\nu_i, g_i) \in C$ so that $\sup_i \tau(\nu_i, g_i) = \infty$.
– there exists a sequence $(\nu_i, g_i) \in C$ so that, as $i \to \infty$, $\nu_i \to \nu$ and g_i converges in the Hausdorff topology to a closed invariant set containing either a nonhyperbolic equilibrium, or more than two orbits.

This definition is inspired by the definition of continuable periodic orbits in one parameter families of vector fields in [AllYor84]. Note that the closure of a homoclinic orbit consists of two orbits. By a closed invariant set containing more than two orbits, one can think of a set containing two homoclinic orbits or a heteroclinic cycle.

Suppose that $(\mu, h) \in G$ is a generically unfolding codimension one homoclinic orbit with $\tau(\mu, h) = \{l(\mu, h)\}$. In such a situation, there is a sequence μ_i of parameter values converging to μ and a periodic orbit h_i of X_{μ_i} converging to h in the Hausdorff topology as $i \to \infty$. For all sufficiently large i, h_i is unique and its unstable manifold $W^u(h_i)$ is either orientable or nonorientable. Note that an unstable manifold which is either one or three dimensional is always orientable. Define

$$\phi(\mu, h) = \begin{cases} 0, & \text{if } W^u(h_i) \text{ is nonorientable for large } i, \\ 1, & \text{if } W^u(h_i) \text{ is orientable for large } i. \end{cases} \tag{4.5}$$

If $W^u(h_i)$ is two dimensional, there exists a two dimensional center manifold $W^{s,u}(h)$ of h [Hom96,San99]. Then $\phi(\mu, h) = 0, 1$ if $W^{s,u}(h)$ is nonorientable, orientable respectively. It is thus possible to define ϕ using X_μ alone.

For any $(\mu, h) \in G$, we let $\phi(\mu, h) = 1$ if the virtual lengths of (μ, h) are bounded and there exists a sequence of families $\{Y_\lambda^i\} \in \mathfrak{X}_2$ with $\{Y_\lambda^i\} \to \{X_\lambda\}$ as $i \to \infty$ and $\{Y_\lambda^i\}$ possesses a generically unfolding homoclinic orbit h_i of codimension one at parameter values μ_i with $\mu_i \to \mu$, $h_i \to h$ in the Hausdorff topology as $i \to \infty$, $\phi(\mu_i, h_i) = 1$. For all other $(\mu, h) \in G$, we let $\phi(\mu, h) = 0$. Denote

$$G_1 = \{(\mu, h) \in G; \;\; \phi(\mu, h) = 1\}. \tag{4.6}$$

The notion of continuability of homoclinic orbits (μ, h) in G_1 is as follows. Let $(\mu, h) \in G_1$ so that h is the union of a homoclinic orbit and a hyperbolic equilibrium. Write Γ_1 for the connected component of G_1 containing (μ, h). We call (μ, h) globally I-continuable if either

– $\Gamma_1 \backslash \{(\mu, h)\}$ is connected

or each component C_1 of $\Gamma_1 \backslash \{(\mu, h)\}$ satisfies at least one of the following conditions:

– C_1 is unbounded
– there exists a sequence $(\nu_i, g_i) \in C_1$ so that $\sup_i \tau(\nu_i, g_i) = \infty$ or so that $(\nu_i, g_i) \to (\nu, g) \in G$ as $i \to \infty$ with (ν, g) possessing unbounded virtual lengths.

– there exists a sequence $(\nu_i, g_i) \in C_1$ so that, as $i \to \infty$, $\nu_i \to \nu$ and g_i converges in the Hausdorff topology to a closed invariant set containing either a nonhyperbolic equilibrium, or more than two orbits.

Theorem 8. [**HomKokNau98**] *Let $(\mu, h) \in G_1$ be a generically unfolding codimension one homoclinic orbit of $\{X_\lambda\}$. Then (μ, h) is globally I-continuable.*

4.1 Pathfollowing homoclinic orbits through infinitely many homoclinic doublings

Here we briefly indicate how the above pathfollowing theory is used to prove the existence of homoclinic doubling cascades in theorem 6.

Let $(x, y) \mapsto \hat{\Phi}_{\mu_1,p,r}(x, y)$ be the rescaled Poincaré return map, see proposition 7. Recall that $\hat{\Phi}_{0,p,r}(x, y) = (\frac{A}{2D}x^\beta, f(x; p, r))$, where

$$f(x; p, r) = p + r(1 - x^\beta)^2. \tag{4.7}$$

Here we have written β for $\beta(0)$. The function f is defined on some interval $(0, A_x]$, for parameters (p, r) from a compact box $I \times J$. A computation shows that $f^2(0; p, r) = 0$ for parameter values on the curve

$$\mathbf{H}_2 = \left\{(p, r);\;\; p + r(1 - (p + r)^\beta)^2) = 0\right\}. \tag{4.8}$$

Observe that $\mathbf{H}_2$ is tangent to the diagonal at $(p, r) = (0, 1)$. Furthermore, along $\{(p, r);\;\; p + r = 2^{1/\beta}\}$, the map f satisfies $f(p + r; p, r) = p + r$.

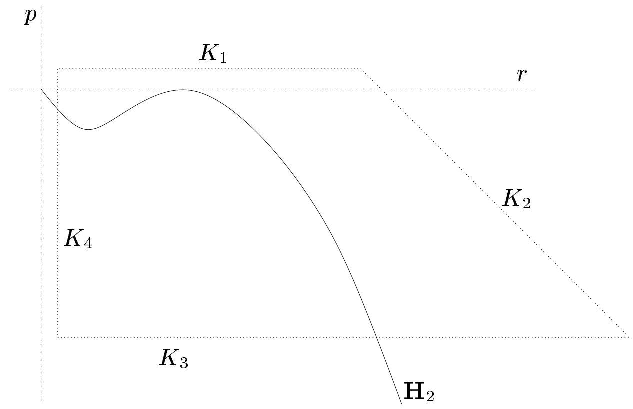


Fig. 3. The choice of the curve K.

We will restrict the parameters (p, r) to a suitably chosen domain in the parameter plane, bounded by a curve K. The curve K consists of four parts

K_1, K_2, K_3, K_4, where for some constants d_1, d_2, P_1, R_1,

$$\begin{aligned}
K_1 &\subset \{(p,r);\ \ p = d_1\},\\
K_2 &\subset \{(p,r);\ \ p + r = 2^{1/\beta} + d_2\},\\
K_3 &\subset \{(p,r);\ \ p = P_1\},\\
K_4 &\subset \{(p,r);\ \ r = R_1\}.
\end{aligned}$$

We take $d_1 > 0$, $d_2 > 0$, $P_1 < 0$ and $R_1 < 1$.

Write X_λ for the vector field depending on $\lambda = (p,r)$. Following notation as above, let $\mu = H_2 \cap K_4$ and let h be the homoclinic orbit of X_μ. Let C_1 be the connected component of $G_1 \backslash \{(\mu,h)\}$ with parameter values that enter the domain bounded by K at μ. Let $\{Y^i_\lambda\}$ be a sequence of generic families converging to $\{X_\lambda\}$ as $i \to \infty$. Generic here means that homoclinic bifurcations unfold generically. For large i, $\{Y^i_\lambda\}$ possesses a homoclinic orbit h_i close to h at parameter values μ_i close to μ. By a combinatorial reasoning one shows the following lemma. Let C^i_1 be defined as C_1 but then for Y^i_λ.

Lemma 9. **[HomKokNau98]** *The closure of* $\{\mu;\ \ (\mu,h) \in C^i_1\}$ *intersects* K *only in* μ_i.

The lemma says that when continuing $\{\mu;\ \ (\mu,h) \in C^i_1\}$ starting at μ_i, into the region bounded by K, one stays in this region. ¿From bifurcation theory and the pathfollowing theorem 8, it follows that $\{Y^i_\lambda\}$ possesses homoclinic orbits of unbounded length along C^i_1. One can now show that also $\{X_\lambda\}$ possesses homoclinic orbits of unbounded length along C_1. ¿From bifurcation theory it is not hard to conclude that this can only happen through a cascade of homoclinic doubling bifurcations; e.g. local and heteroclinic bifurcations can not occur since the dynamics is confined to a tubular neighborhood of the degenerate homoclinic orbit.

5 Renormalization theory

A striking aspect of period doubling cascades is its universality; independent of details of the system, the bifurcation values scale with a fixed geometric rate. This universality was explained by renormalization theory, see [CouTre78,Fei78]. We will indicate the universal scaling structures found in the bifurcation diagram of homoclinic doubling cascades and discuss its explanation through renormalization theory. We treat renormalization for the model interval maps, it is expected that the vector fields show the same universal scalings in their bifurcation diagrams.

Let $\mathfrak{C}^\omega$ denote the set of functions $x \mapsto f(x)$, defined on a compact interval $[0, L]$, that are of the form $f(x) = g(x^\beta)$ for a real analytic function $u \mapsto g(u)$, and that are unimodal with a unique quadratic minimum at $x = 1$. We require that g extends to a bounded analytic function on a fixed complex neighborhood Ω of $[0, L^\beta]$. For $f \in \mathfrak{C}^\omega$ with $f(x) = g(x^\beta)$, let

$$\|f\|_0 = \sup_{u \in \Omega} |g(u)|.$$

Equipped with this norm, $\mathfrak{C}^\omega$ is a cone in a Banach space.

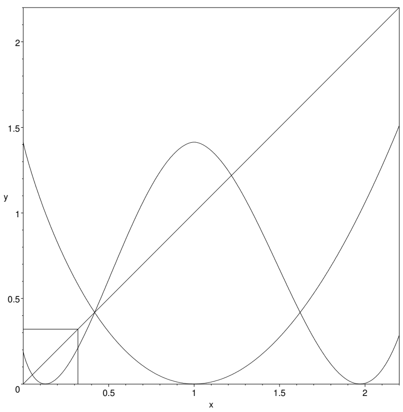


Fig. 4. The graphs of f and f^2 for $f(x) = r(1-x^\beta)^2$ with $\beta = 0.9$ and $r = 1.413178$.

We can write a function $f \in \mathfrak{C}^\omega$ as

$$f(x) = p + r(x^\beta)(1 - x^\beta)^2,$$

where p is a real number and $u \mapsto r(u)$ is a positive C^ω function. We are interested in such functions for which β is close to $\frac{1}{2}$, and for which p is small and r is close to 1 on a compact interval. Write

$$r(u) = 1 + \varepsilon(u).$$

The renormalization $\mathcal{R}(f)$ of f is a rescaling of the second iterate f^2. It will be defined for functions f close to $x \mapsto (1 - x^\beta)^2$ for which $f(0) > 1$. There exists $a \in (0, 1)$ for which $f(a) = 1$. Let

$$\mathcal{R}(f)(x) = \frac{1}{a} f^2(ax).$$

The condition $f(0) > 1$ is equivalent to $p > -\varepsilon(0)$. The rescaling factor a is such that $\mathcal{R}(f)$ has its critical point at $x = 1$, so that $\mathcal{R}(f) \in \mathfrak{C}^\omega$.

The following theorem provides an isolated fixed point of the renormalization operator $\mathcal{R}$.

Theorem 10. [**HomYou99**] *For $\beta > \frac{1}{2}$ and $\beta - \frac{1}{2}$ small, the renormalization operator $\mathcal{R}$ on $\mathfrak{C}^\omega$ possesses a hyperbolic fixed point ϕ. The function ϕ depends continuously on β and converges to $x \mapsto (1-\sqrt{x})^2$ as $\beta \to \frac{1}{2}$. The linearization $D\mathcal{R}$ at ϕ has two unstable eigenvalues δ_1, δ_2. They depend continuously on β and satisfy $\delta_1 \to 2$, $\delta_2 \to \infty$ as $\beta \to \frac{1}{2}$. The remainder of the spectrum of $D\mathcal{R}(\phi)$ is strictly inside the unit disc. The unstable manifold of $\mathcal{R}$ is 2 dimensional.*

5.1 Universal scalings

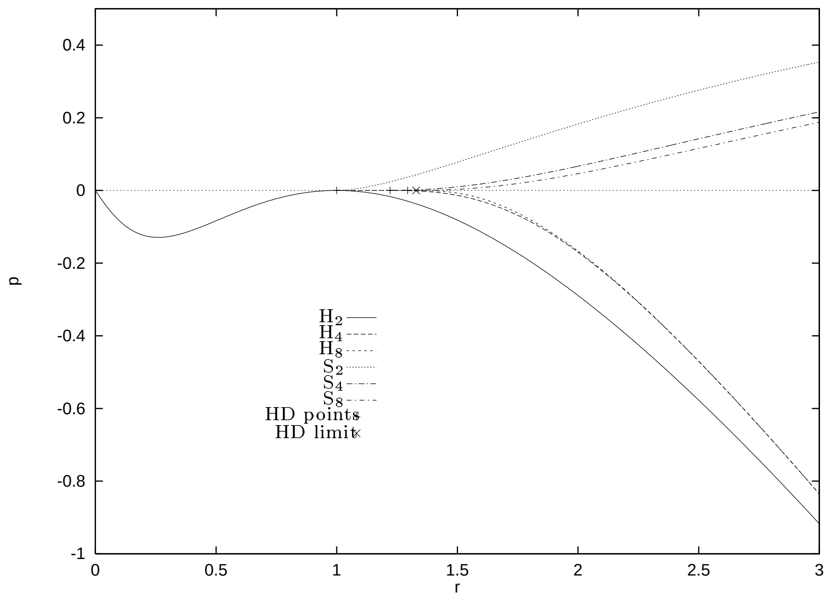


Fig. 5. Bifurcation curves for $x \mapsto p + r(1-x^\beta)^2$ in the (p,r) parameter plane for $\beta = 0.6$.

Theorem 10 provides an explanation of universal scalings in the bifurcation diagram of generic two parameter families of functions.

It follows from invariant manifold theory that $\mathcal{R}$ possesses local stable and unstable manifolds near ϕ, which are denoted by $W^s(\phi)$ and $W^{u,uu}(\phi)$, respectively. The local unstable manifold contains a strong unstable manifold $W^{uu}(\phi)$ and the weak unstable manifold $W^u(\phi) = W^{u,uu}(\phi) \cap \mathfrak{C}_0^\omega$. Note that $W^u(\phi)$ is a smooth manifold.

The next theorem is basic in finding scaling structures in bifurcation diagrams.

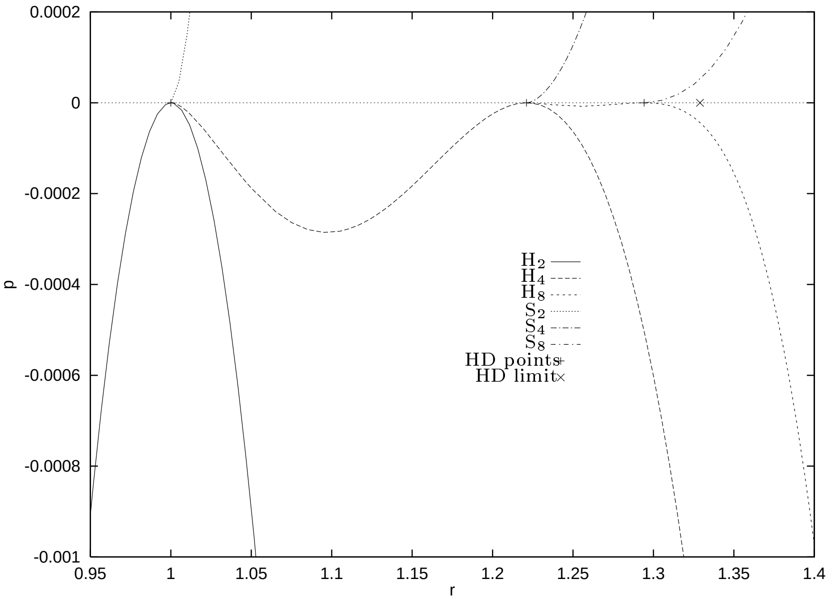


Fig. 6. Blow up of figure 5.

Theorem 11. [**HomYou99**] *Consider a two parameter family* $\{f_{p,\varepsilon};\ (p,\varepsilon) \in \mathbb{R}^2\}$ *of functions in* $\mathfrak{C}^\omega$ *with* $f_{0,\varepsilon} \in \mathfrak{C}_0^\omega$, *that intersects the local stable manifold* $W^s(\phi)$ *transversally at some function* $f_{0,\bar{\varepsilon}}$. *Let* (p_n, ε_n) *be a sequence of parameter values tending to* $(0, \bar{\varepsilon})$, *such that* $\mathcal{R}^{2^n}(f_{p_n,\varepsilon_n})$ *converges to a function* $\bar{f}$ *in the local unstable manifold* $W^{u,uu}(\phi)$. *Then*

$$\frac{p_{n+1} - p_n}{p_n - p_{n-1}} \to \frac{1}{\delta_2} \quad \text{if } \bar{f} \notin W^u(\phi),$$
$$\frac{\varepsilon_{n+1} - \varepsilon_n}{\varepsilon_n - \varepsilon_{n-1}} \to \frac{1}{\delta_1} \quad \text{if } \bar{f} \notin W^{uu}(\phi).$$

The theorem can be applied to sequences of parameter values (p_n, ε_n) so that $\mathcal{R}^{2^n}(f_{p_n,\varepsilon_n})$ is contained in a bifurcation surface (e.g. H_2, HD_2, or S_2) and converges to a function $\bar{f}$ in the intersection of this bifurcation surface with $W^{u,uu}(\phi)$. The theorem shows that scalings exist in bifurcation diagrams of generic two parameter families of functions in $\mathfrak{C}^\omega$, independent of details of the family, other than the value of β. It relates the unstable eigenvalues at the fixed point ϕ of the renormalization operator $\mathcal{R}$ to scalings in such bifurcation diagrams.

In figures 5 and 6, bifurcation curves for the family $x \mapsto p + r(1 - x^\beta)^2$ with $\beta = 0.6$ are shown. These figures illustrate the scalings predicted by Theorem 11. In particular, the HD points converge exponentially at a rate

δ_1 which is close to 2 for β close to $\frac{1}{2}$. As is illustrated in figure 6, the curve pieces in $\mathrm{H}_{2^{n+1}}$ between HD_{2^n} and $\mathrm{HD}_{2^{n+1}}$, accumulate on the line $\{p = 0\}$ exponentially fast at the strong rate δ_2.

It should be noted that Theorem 11 does not explain all scalings that exist in the bifurcation diagram. To see this, define $\mathrm{H}_\infty = \lim_{k\to\infty} \mathrm{H}_{2^k}$, note that H_∞ is invariant under $\mathcal{R}$. Similarly, $\mathrm{P}_\infty = \lim_{k\to\infty} \mathrm{S}_{2^k}$ is invariant under $\mathcal{R}$. H_∞ and P_∞ constitute a 'boundary of chaos'. Theorem 11 does not address scalings of sequences of bifurcation values in S_{2^n} converging to a point in P_∞ away from $W^s(\phi)$ (which are expected to scale exponentially with the usual Feigenbaum rate). Nor does it address scalings of sequences of bifurcation values in H_{2^n} converging to a point in H_∞ away from $W^s(\phi)$. We have no information on these scalings (however, [LyuPikZak89] and [Hom96] contain results on scalings of homoclinic bifurcations in similar situations).

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