Topological and Measurable Dynamics of Lorenz Maps

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Abstract We investigate the dynamics of Lorenz maps, in particular the asymptotical behaviour of the trajectory of typical points. For Lorenz maps f with negative Schwarzian derivative we give a classification of the possible metric attractors and show that either f has an ergodic absolutely continuous invariant probability measure of positive entropy or the iterates of typical points spend most of their time shadowing the trajectory of one of the two critical values. Our main tool therefore is the construction of Markov extensions for Lorenz maps which provide a unified framework to approach both the topological and the measurable aspects of the dynamics.

We study the bifurcation diagram of a smooth two parameter family of Lorenz maps which describes the parameter dependence of the kneading invariant and show that essentially every admissible kneading invariant actually occurs if the family is sufficiently rich. Finally, we adress the problem whether the kneading invariant depends monotonously on the parameters.

1 Introduction

Lorenz maps play an important role in the study of the global dynamics of families of vector fields near homoclinic bifurcations. A typical situation where Lorenz maps are encountered is given by a family $X_\mu : \mathbb{R}^3 \to \mathbb{R}^3$ of vector fields, where X_0 has a hyperbolic saddle with eigenvalues $\lambda_{ss} < \lambda_s < 0 < \lambda_u$ and with two homoclinic orbits in the configuration of a butterfly connecting the stable and unstable direction, like in the classical Lorenz system [25]. Breaking up the homoclinic loops by changing the parameters it is possible to find vector fields with very complicated chaotic dynamics, where the trajectories of points seem to randomly follow one of two loops near the former homoclinic orbits.

The geometric Lorenz attractor

To explain the dynamics on the “strange” attractor of the Lorenz system, Guckenheimer [8] proposed a two-dimensional model for the flow on the attractor, the so-called *geometric Lorenz attractor*. It is obtained, roughly speaking, by forgetting about the strong stable direction and consists of a two-dimensional branched manifold with a hyperbolic saddle as it is shown in Figure 1. Considering the Poincaré map to the cross section $\Sigma = [p, q]$

one obtains an interval map with two monotonous branches: If c denotes the intersection of Σ with the stable manifold of the saddle then all points to the left of c follow the left loop and all points to the right of c follow the right loop until they hit the cross section again.

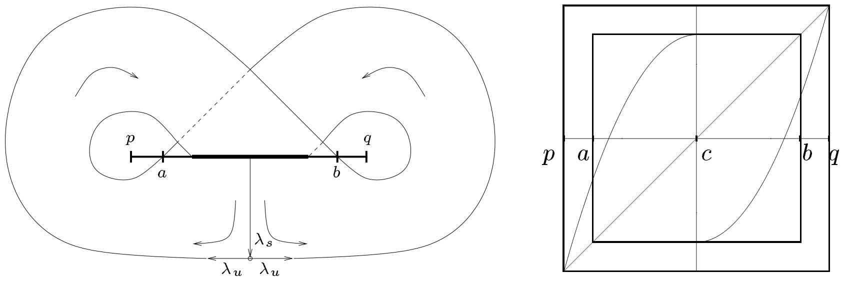

Figure1. The left hand side shows a phase portrait of the flow on the branched manifold and the right hand side the first return map to the cross section $\Sigma = [p, q]$.

Hence from a topological viewpoint a Lorenz map is nothing else than an interval map with two monotonous and continuous branches and a discontinuity in between. From a metrical viewpoint it is smooth to some degree on both branches and of order $\alpha > 0$ at the discontinuity, i.e., $f(c \pm \varepsilon) \sim f(c^{\pm}) \pm \varepsilon^{\alpha}$, where the order originates from the local analysis of the linearized flow at the saddle and equals the ratio $\frac{|\lambda_s|}{|\lambda_u|}$ of the stable and the unstable eigenvalue. In practice, the smoothness of the Lorenz map is limited by the fact that in order to rigorously justify the geometric Lorenz model, the dynamics of the vector field has to be reduced by means of geometrical methods like the construction of invariant foliations or the existence of a two-dimensional invariant centre manifolds. Even if this can be done, the smoothness of the foliation or the centre manifold depends on certain gap conditions on the eigenvalues of the vector field. In the worst case it can happen that it is only of class $\mathcal{C}^{1+\varepsilon}$, i.e., once differentiable with Hölder continuous derivative (cf. Homburg [17] for an example). Nevertheless, for some metrical results we assume that it is at least of class $\mathcal{C}^2$.

If $\alpha < 1$ then the derivative of f is infinite at the discontinuity. Such maps are typically overall expanding and chaotic. Their topological type is completely determined by the *kneading invariant* of the map, a pair of binary sequences coding the orbits of the two *critical points* c^+ and c^-, (i.e., the corresponding one-sided limits of the orbits at the discontinuity c). Moreover, the set of all possible kneading invariants can be characterized by a simple combinatorial condition. Since $\alpha < 1$ holds in the situation of the classical Lorenz system, this kind of Lorenz maps has been studied by many people and their dynamics is well understood, see for example Guckenheimer [8], Rand [31], Guckenheimer & Williams [9], Williams [36], Parry [30], Hubbard & Sparrow [18], and Glendinning & Sparrow [7].

Here we are mainly concerned with Lorenz maps of exponent $\alpha > 1$. One reason for this is that in this case the derivative of f vanishes at the discontinuity which means that such maps are typically contracting in some regions and expanding in others, and due to the interplay between contraction and expansion such Lorenz maps can exhibit a much wider spectrum of behaviour. Another reason is that the condition $\alpha > 1$ is compatible with the condition of negative Schwarzian derivative,(1) whereas the condition $\alpha < 1$ is not. Under the assumption of negative Schwarzian derivative one has strong tools to control the distortion on branches of high iterates of the map, just as for smooth interval maps.

Although the theory of vector fields serves as the motivation for studying Lorenz maps, we focus here on what can be said about Lorenz maps from the viewpoint of one-dimensional dynamics. That is, we study the time discrete dynamical system (X, f), where $f : X \to X$ is a Lorenz map defined on some compact interval $X \subset \mathbb{R}$. Given an initial point $x \in X$, its time evolution under the action of f is described by the *orbit* or *trajectory* of x,

$$x_0 \to x_1 \to x_2 \to x_3 \to \dots \qquad \text{, where } x_n := f^n(x) \; .$$

The asymptotic behaviour of typical points

A near at hand way to describe the asymptotic behaviour of the trajectory of a point x is to look at the set of its accumulation points, called the *ω-limit set* of x. In general, different points $x \in X$ can have very different asymptotic behaviour. From a physical viewpoint, a certain type of asymptotic behaviour is relevant only if it is observable with positive probability, which means that it is exhibited by a set of initial conditions which has positive Lebesgue measure. This opinion is reflected by the following definition of an attractor which goes back to Milnor [27].

Definition 1 (Attractor). A set $A \subseteq X$ is called *f-invariant* if $f(A) \subseteq A$. For a compact f-invariant set $A \subseteq X$ let $B(A) := \{x \in X \mid \omega(x) \subseteq A\}$ denote the *basin of attraction* of A. Then A is called an *attractor* if the following holds.

1. A attracts a set of positive Lebesgue measure: $m(B(A)) > 0$.
2. Every proper subset $\tilde{A}$ of A which is compact and f-invariant attracts significantly less points: $m(B(A) \setminus B(\tilde{A})) > 0$.

An attractor A is called the *global attractor* if $B(A) = X \bmod m$. It is called *indecomposable* if it is not the disjoint union of two smaller attractors and *minimal* if it does not contain any smaller attractors.

(1) For $f \in \mathcal{C}^3$ the Schwarzian derivative is defined as $Sf := \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$ where $f' \neq 0$.

The question about the asymptotic behaviour of typical points can now be posed as follows: What is the global attractor of f and can it be decomposed into minimal attractors? The answer as given in Theorem 16 can be summarized as follows:

- ▹ For Lorenz maps with negative Schwarzian derivative there is always a global attractor. It is minimal unless the map has two attracting periodic orbits, in which case it splits up into two minimal attractors.

A different kind of information about the asymptotic behaviour of a point can be obtained if one looks at the empirical distributions of trajectories instead of their ω-limit sets. More precisely, one considers the empirical distributions $\frac{1}{n}\sum_{k=0}^{n-1}\delta_{x_k}$, where δ_y denotes the Dirac measure in y, and asks whether they converge weakly to some f-invariant measure μ, or more generally, what the set of weak limit points is. Again, the physically most interesting measures are the ones that can be observed for a positive measured set of initial conditions.

Definition 2 (SRB-measure). A probability measure μ is called a *Sinai-Ruelle-Bowen measure*, or just an *SRB-measure*, if the set $\{x \in X \mid \mu_n(x) := \frac{1}{n}\sum_{k=0}^{n-1}\delta_{x_k} \to \mu\}$ has positive Lebesgue measure.

A trivial example of an SRB-measure is the equidistribution on an attracting periodic orbit. Much more interesting is the case where f has an absolutely continuous invariant probability measure which is ergodic on its support. In this case $\mu_n(x) \to \mu$ holds μ-almost surely by the Ergodic Theorem, whence on a set of positive Lebesgue measure.

For Lorenz maps with negative Schwarzian derivative it turns out that there are essentially two different possibilities which can be discriminated by means of upper Lyapunov-exponents $\lambda^+(x) := \limsup_{n\to\infty}\frac{1}{n}\log|(f^n)'(x)|$.

- ▹ Either $\lambda^+(x) > 0$ on a set of full Lebesgue measure. Then f has a unique (and hence ergodic) absolutely continuous invariant probability measure of positive entropy (Theorem 18).
- ▹ Or $\lambda^+(x) \le 0$ on a set of full Lebesgue measure. In this case all weak accumulation points of the sequences $(\mu_n(x))_n$ are contained in the convex closure of the union of the weak accumulation points of $(\mu_n(c^+))_n$ and $(\mu_n(c^-))_n$ for Lebesgue-almost every x (Theorem 19).

The topological conjugacy class of a Lorenz map is essentially determined by the kneading sequences of c^+ and c^-. These are 0-1-sequences which encode the topological dynamics of the map. In Theorem 22 those 0-1-sequences that can occur as kneading sequences of Lorenz maps are characterized in various purely combinatorial ways. This characterization is one of the ingredients of a bifurcation analysis for $\mathcal{C}^1$-Lorenz maps that can be summarized as follows:

- ▹ Essentially all kneading sequences can be realized within any given $\mathcal{C}^1$-Lorenz family with critical order $\alpha \ge 2$, naturally parametrized by two parameters (Theorem 25).

▹ The finite initial pieces of the kneading sequences provide, in a natural way, a bifurcation diagram for $\mathcal{C}^1$-Lorenz families in their two-dimensional parameter space (Theorem 28 and Figure 4).

In the following we give an outline of the results obtained in [33]. Unless stated otherwise, all the theorems below are taken from there.

2 Markov extensions

Our main tool to study the dynamics of Lorenz maps is the construction of *Markov extensions*. The great advantage of Markov extensions is that they provide a unified framework for the topological and measurable aspects of the dynamics.

The Markov diagram

Markov extensions for piecewise monotonous interval maps were originally introduced by Hofbauer [10,11,13,12,14,15] as countable state topological Markov chains, which he called *Markov diagrams*. The state space of the Markov chain is a collection $\mathcal{D}$ of subintervals $D \subset X$ which are constructed in such a way that every interval $D \in \mathcal{D}$ is mapped by f onto the union of finitely many intervals from $\mathcal{D}$, called the *successors* of D. The successor relation defines the possible transitions for $\mathcal{D}$. Every point from X can be represented by a path in the Markov diagram which symbolically codes the sequence of intervals that are visited by the iterates of x. In this way the original system (X, f) becomes a continuous factor of the Markov chain. This fact can be used to derive statements about topological properties of f like the nature of the nonwandering set or the topological entropy.

Hofbauer towers and Markov extensions

Later Keller [19–23] turned the Markov diagram into an even more powerful tool by taking into account that its "states" are more than that, namely intervals which carry a smooth structure and a canonical measure, the Lebesgue measure. He introduced the so called *(canonical) Markov extension* of f, a piecewise smooth dynamical system $(\hat{X}, \hat{f})$ which has a countable Markov partition and a natural projection $\pi : \hat{X} \to X$ yielding the original system (X, f) as a countable-to-one factor. Let us describe his construction of the Markov extension in more detail.

Definition 3. i) Let $\mathcal{Z} := \{Z^-, Z^+\} := \{(p, c), (c, q)\}$ denote the partition of X into the intervals of monotonicity of f and let $\mathcal{D} \supseteq \mathcal{Z}$ be the smallest family of intervals satisfying

$$f(D \cap Z^\pm) \in \mathcal{D} \quad \text{whenever} \quad D \in \mathcal{D} \text{ and } D \cap Z^\pm \neq \emptyset \,.$$

Now take a collection $\hat{\mathcal{D}} := \{\hat{D} \mid D \in \mathcal{D}\}$ of disjoint copies $\hat{D} := D \times \{D\}$ of the intervals $D \in \mathcal{D}$. Their union $\hat{X} := \bigcup_{\hat{D} \in \hat{\mathcal{D}}} \hat{D}$ is called the *Hofbauer tower* associated to the map f and the sets $\hat{D}$ are called the *levels* of the tower.

ii) The *Markov extension* for f is the dynamical system $(\hat{X}, \hat{f})$ where $\hat{f} : \hat{X} \to \hat{X}$ is defined in the following way:

$$\hat{f}(\hat{x}) := (f(x), f(D \cap Z[x])) \qquad \text{for } \hat{x} = (x, D) \in \hat{D}.$$

Here $Z[x]$ denotes the element of the partition $\mathcal{Z}$ containing x. The natural projection $\pi : \hat{X} \to X$ is given by $\pi(\hat{x}) := x$ for $\hat{x} = (x, D) \in \hat{D}$ and satisfies $\pi \circ \hat{f} = f \circ \pi$. The value $\hat{f}(\hat{c})$ is left undefined for all *critical points* $\hat{c} \in \pi^{-1}(c)$.

Note that the tower depends on the map f, although this is not indicated by the notation. The terms *tower* and *level* originate from the fact that one can imagine all these intervals piled up one upon the other, in this way forming the floors of an infinite tower built on the base $X = (p, q)$, and the map π as the vertical projection from the tower onto its base.

The possible transitions of points in the tower with respect to this Markov partition are described by Hofbauers Markov diagram.

Definition 4. The *Markov diagram* for $(\hat{X}, \hat{f})$ is the directed graph $(\hat{\mathcal{D}}, \to)$ with edges given by the following relation:

$$\hat{C} \to \hat{D} :\Longleftrightarrow \hat{D} \subseteq \hat{f}(\hat{C}) \qquad\qquad (\hat{C}, \hat{D} \in \hat{\mathcal{D}})$$

If $\hat{C} \to \hat{D}$ then $\hat{D}$ is called a *successor* of $\hat{C} \in \hat{\mathcal{D}}$.

Cutting times and the kneading map

In the situation of Lorenz maps it turns out that the Markov diagram allows only a very limited set of transitions: The levels can be grouped into two different sets, denoted by $\hat{\mathcal{D}}^+ := \{\hat{D}_n^+ \mid n \in \mathbb{N}\}$ and $\hat{\mathcal{D}}^- := \{\hat{D}_n^- \mid n \in \mathbb{N}\}$, such that for every level $\hat{D}_n^\pm$ there is a transition $\hat{D}_n^\pm \to \hat{D}_{n+1}^\pm$ and for some special levels—called the *critical levels*—there is a second arrow $\hat{D}_n^\pm \to \hat{D}_{\tilde{n}+1}^\mp$ where $\hat{D}_{\tilde{n}}^\mp$ is a particular critical level $\hat{D}_{\tilde{n}}^\mp$ on the other side with $\tilde{n} < n$.[(2)] Graphically this means that one can either *climb up* one level in the tower or *jump down* from a critical level to a lower or equal level on the other side of the tower.

The indices of the critical levels are called the *cutting times* and are numbered $(S_k^\pm)_{k \geqslant 0}$ in increasing order. The Markov diagram is completely determined by the *kneading map* Q.

(2) Here and in the following all statements and expressions containing the symbols "$\pm$" and "$\mp$" are intended to be read twice, once using the upper signs and once using the lower signs simultaneously.

Proposition 5. *There exists a map* $Q = (Q^+, Q^-) : \mathbb{N} \to \mathbb{N}_\infty \times \mathbb{N}_\infty$ *such that*

$$S_k^{\pm} - S_{k-1}^{\pm} = S_{Q^\pm(k)}^{\mp} \qquad \textit{for all } k \geqslant 1.$$

The map Q *is called the* kneading map *of* f.

A characteristic feature of the kneading map is the so called *Hofbauer condition*, a combinatorial condition which is stated in Theorem 22 below.

Decomposition of the Markov diagram

The information about the combinatorial peculiarities of the Markov diagram can be used to prove the following decomposition into irreducible components. Let us introduce some notation for the following theorem: Write $\hat{C} \twoheadrightarrow \hat{D}$ if there is a chain of transitions connecting $\hat{C}$ with $\hat{D}$. An *irreducible component* $\hat{\mathcal{X}}$ is a maximal collection of levels such that $\hat{C} \twoheadrightarrow \hat{D}$ holds for all $\hat{C}, \hat{D} \in \hat{\mathcal{X}}$ and a level $\hat{D}$ is called *transient* if $\hat{D} \not\twoheadrightarrow \hat{D}$.

Theorem 6. *The Markov diagram* $(\hat{\mathcal{D}}, \to)$ *of a Lorenz map can be decomposed as follows: The ground levels* $\hat{D}_1^+$ *and* $\hat{D}_1^-$ *are always transient. All other levels can be grouped into a finite chain* $\hat{\mathcal{T}}_1 < \hat{\mathcal{X}}_1 < \hat{\mathcal{T}}_2 < \hat{\mathcal{X}}_2 < \cdots < \hat{\mathcal{X}}_m < \hat{\mathcal{T}}_{m+1}$ *or an infinite chain* $\hat{\mathcal{T}}_1 < \hat{\mathcal{X}}_1 < \hat{\mathcal{T}}_2 < \hat{\mathcal{X}}_2 < \dots$ *of disjoint subsets of* $\hat{\mathcal{D}} \setminus \{\hat{D}_1^+, \hat{D}_1^-\}$ *which are of the following type:*

$\hat{\mathcal{X}}_i$ *is an irreducible component*
$\hat{\mathcal{T}}_i$ *is either void or contains only transient levels*

The ordering signifies that there are paths traversing the components from left to right but no paths in the other direction.

Renormalization

We introduce two types of renormalizations for Lorenz maps, *proper* and *nonproper* renormalizations. Roughly speaking, a Lorenz map is (n, m)-*renormalizable* if one can find branches of some iterates f^n and f^m to the left and to the right of the critical point, respectively, such that the restriction of those two branches to the interval $[f^m(c^+), f^n(c^-)]$ looks like a Lorenz map which has been restricted to its dynamical interval. If the two branches extend to a larger interval $[\tilde{p}, \tilde{q}]$ such that the restriction to $[\tilde{p}, \tilde{q}]$ looks like a complete Lorenz map again then the renormalization is *proper* (compare Figures 2 and 1, rhs.).

There is a close relation between the above decomposition of the Markov diagram and the possible renormalizations for the Lorenz map. Before stating the following theorem we remark that in the properly renormalizable case both branches are obviously critical (i.e., $m = S_k^+$ and $n = S_l^-$ for some indices k and l) and that the same can be assumed w.l.o.g. in the nonproperly renormalizable case if f has no periodic attractor.

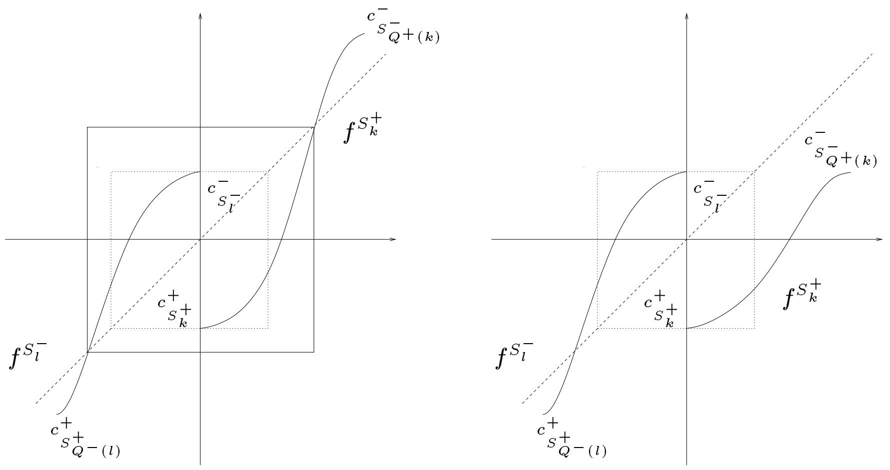

Figure2. A properly and a nonproperly renormalizable Lorenz map

Theorem 7. *Assume that the Lorenz map f has no periodic attractor and let $(\hat{\mathcal{D}}, \rightarrow)$ be its Markov diagram. Then the following holds.*

1. *f is (S_k^+, S_l^-)-renormalizable iff the set $\hat{\mathcal{D}}^{\wedge}_{k\,l} := \{\hat{D}_i^+, \hat{D}_j^- \mid i > S_k^+, j > S_l^-\}$ of levels above $\hat{D}^+_{S_k^+}$ respectively $\hat{D}^-_{S_l^-}$ is invariant in the Markov diagram.*
2. *f is properly (S_k^+, S_l^-)-renormalizable iff the set $\hat{\mathcal{D}}^{\wedge}_{k\,l}$ is invariant and the levels $\hat{D}^+_{S_k^+}$ and $\hat{D}^-_{S_l^-}$ belong to an irreducible component $\hat{\mathcal{X}}_i$ immediately below $\hat{\mathcal{D}}^{\wedge}_{k\,l}$.*

The kneading map $\mathcal{R}Q$ of the renormalized map is given by "shifting" the kneading map Q, i.e., $\mathcal{R}Q^+(j-k) = Q^+(j) - l$ for $j > k$ and $\mathcal{R}Q^-(j-l) = Q^-(j) - k$ for $j > l$.

Stated briefly, this means that proper renormalizations of f correspond to the irreducible components of the Markov diagram and nonproper renormalizations of f correspond to the transient parts in between.

3 Hopf decompositions and attractors

Transfer operators

The smooth structure on the Markov extension makes it possible to study the time evolution of densities of absolutely continuous measures under the influence of $\hat{f}$ by means of the *transfer operator* $\mathrm{P}_{\hat{f}}$ (also known as the *Perron-Frobenius operator*) on the tower. It is defined with respect to a reference measure $\hat{m}$ which is the natural lift of the Lebesgue measure m onto the levels of the tower via the projection π.[3]

Definition 8. The *transfer operator* for $\hat{f}$ is the map $\mathrm{P}_{\hat{f}} : \mathrm{L}^1_{\hat{m}}(\hat{X}) \to \mathrm{L}^1_{\hat{m}}(\hat{X})$ defined by $\hat{f}(\hat{\psi} \cdot \hat{m}) = \mathrm{P}_{\hat{f}}(\hat{\psi}) \cdot \hat{m}$. Its explicit form is given by

$$\mathrm{P}_{\hat{f}}(\hat{\psi}) = \sum_{\hat{Z} \in \hat{\mathcal{Z}}} \mathrm{P}_{\hat{f}}(\hat{\psi}_{\hat{Z}}) = \sum_{\hat{Z} \in \hat{\mathcal{Z}}} (\hat{\psi} \cdot \hat{\phi}) \circ \hat{f}^{-1}_{\hat{Z}} \ ,$$

where $\hat{\psi}_{\hat{Z}} := \hat{\psi} \cdot 1_{\hat{Z}}$, $\hat{\phi} := \frac{1}{|\hat{f}'|}$ and $(\hat{\psi} \cdot \hat{\phi}) \circ \hat{f}^{-1}_{\hat{Z}} := (\hat{\psi} \cdot \hat{\phi})|_{\hat{Z}} \circ (\hat{f}|_{\hat{Z}})^{-1} \cdot 1_{\hat{f}(\hat{Z})}$. Using its explicit form the transfer operator can also be defined for arbitrary nonnegative measurable functions. When talking of *densities* or *mass distributions* we mean such nonnegative measurable functions, not necessarily integrable.

Similarly, from the original map $f : X \to X$ and the projection $\pi : \hat{X} \to X$ one obtains transfer operators $\mathrm{P}_f : \mathrm{L}^1_m(X) \to \mathrm{L}^1_m(X)$ and $\mathrm{P}_\pi : \mathrm{L}^1_{\hat{m}}(\hat{X}) \to \mathrm{L}^1_m(X)$, respectively. The operators P_f and $\mathrm{P}_{\hat{f}}$ are related by $\mathrm{P}_f \circ \mathrm{P}_\pi = \mathrm{P}_\pi \circ \mathrm{P}_{\hat{f}}$. By construction, the densities of absolutely continuous invariant measures are precisely the fixed points of the transfer operator: $\hat{f}(\hat{h} \cdot \hat{m}) = \hat{h} \cdot \hat{m}$ if and only if $\hat{\mathrm{P}}(\hat{h}) = \hat{h}$, and $f(h \cdot m) = h \cdot m$ if and only if $P(h) = h$. Moreover, if $\hat{h}$ is an invariant probability density for $\hat{f}$ then $h := \mathrm{P}_\pi(\hat{h})$ is an invariant probability density for f.

Regular densities

As in the theory of countable state Markov chains the topological Markov structure already yields a lot of information about the existence and location of absolutely continuous invariant measures, since the mass can only be transported along the allowed transitions in the tower. Additionally, due to the non-linearity of the map $\hat{f}$, the mass will be compressed in some parts of the tower and thinned out in other parts. How big this effect is depends on the distortion of the map. In order to control the distortion we assume that the map has negative Schwarzian derivative, i.e., that f is a $\mathcal{C}^3$-Lorenz map with $f' > 0$ and $Sf := \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 < 0$ on both sides of the discontinuity.

Let us briefly state some properties of maps with negative Schwarzian derivative. From the chain rule for the Schwarzian derivative[4] it follows that iterates of maps with negative (resp. positive) Schwarzian derivative again have negative (resp. positive) Schwarzian derivative. A straightforward calculation yields the equality $(|f'|^{-\frac{1}{2}})'' = -\frac{1}{2}|f'|^{-\frac{1}{2}} Sf$ which shows that

(3) Its explicit form is given by $\hat{m}(\hat{A}) := \sum_{\hat{D} \in \hat{\mathcal{D}}} m(\pi(\hat{A} \cap \hat{D}))$ for every measurable set $\hat{A} \subseteq \hat{X}$.

(4) The chain rule for the Schwarzian derivative is $S(g \circ f) = Sg \circ f \cdot |f'|^2 + Sf$.

$Sf < 0$ (resp. $Sf > 0$) if and only if $|f'|^{-\frac{1}{2}}$ is strictly convex (resp. strictly concave). From this one immediately obtains the *Koebe Principle* which is an important tool to estimate the distortion on the branches of iterates of f.

Lemma 9 (Koebe Principle). *Let $I \subseteq Z$ be two intervals and let $g : Z \to g(Z)$ be a diffeomorphism with negative Schwarzian derivative. If $g(Z)$ contains a δ-scaled neighbourhood of $g(I)$, i.e., if the length of both components of $g(Z) \setminus g(I)$ is at least $\delta \cdot |g(I)|$, then the distortion $\sup_{x,y\in I} |f'(x)/f'(y)|$ on I is bounded by $\left(\frac{1+\delta}{\delta}\right)^2$.*

The proof of the Koebe Principle is elementary[5] and based on the fact that $h := g^{-1}$ has positive Schwarzian derivative by the chain rule, whence $|h'|^{-\frac{1}{2}}$ is strictly concave on $g(Z)$. Intuitively, the Koebe Principle states that maps with negative Schwarzian derivative behave very much like linear maps on every branch if one stays away far enough from the ends of the branch, i.e., if there is some "Koebe space". That makes it sound reasonable that densities are not distorted too much when transported along such branches—at least this holds for the part of the mass that is located away from the boundary of the branch. Indeed, Keller [21] showed using results of Misiurewicz [29] that for maps with negative Schwarzian derivative there is a large positive cone of nice *regular densities* $\hat{\psi}$ which is invariant by the transfer operator and for which the quotients $\hat{\psi}(\hat{x})/\hat{\psi}(\hat{y})$ are bounded uniformly in $\hat{\psi}$ on every compact subset of the tower.

Proposition 10. *i) $\mathcal{H}$ is a positive cone which is closed in the compact open topology and $\mathcal{H} - \mathcal{H} = \mathcal{C}(\hat{X})$ is dense in $\mathrm{L}^1_{\hat{m}}(\hat{X})$. The cones $\mathcal{H} \cap \mathrm{L}^1_{\hat{m}}(\hat{X})$ and $\mathcal{H}_2 \cap \mathrm{L}^1_{\hat{m}}(\hat{X})$ are invariant by $\hat{\mathrm{P}}$.*

ii) For every $\delta > 0$ there exists a constant $C(\delta) > 0$ with the following property: Whenever $\hat{D} \in \hat{\mathcal{D}}$ contains a δ-scaled neighbourhood of $\hat{J} \subset \hat{D}$ and $\hat{\psi} \in \mathcal{H}$ is positive on $\hat{D}$ then

$$\left| \frac{\hat{\psi}(\hat{x})}{\hat{\psi}(\hat{y})} - 1 \right| \leqslant \frac{C(\delta)}{|\hat{J}|} \cdot |\hat{x} - \hat{y}| \quad \text{for all } \hat{x}, \hat{y} \in \hat{J}.$$

In particular, for every compact set $\hat{K} \subset \hat{X}$ there is a uniform bound for the Lipschitz norm of the family $\left\{ \log \hat{\psi}|_{\hat{K}} \;\middle|\; \hat{\psi} \in \mathcal{H},\, \hat{\psi}|_{\hat{K}} > 0 \right\}$ which depends only on the relative Koebe space around $\hat{K}$ in $\hat{X}$.

The Hopf Decomposition

As a consequence of the distortion bounds the transfer operator on the Markov extension behaves very much like a countable state stochastic Markov chain and it is possible to derive strong statements about the conservative

[5] A much more general version of the Koebe Principle is proved in [5, Section IV.1].

and dissipative part and the existence of absolutely continuous invariant measures for $\hat{f}$. Before stating the results let us briefly recall some facts about the Hopf decomposition and introduce some notation (cf. Krengel [24]).

A measurable set $\hat{A} \subset \hat{X}$ is called a *wandering set* if $\hat{f}^n(\hat{A}) \cap \hat{A} = \emptyset$ for all $n > 0$. The map $\hat{f}$ is called *conservative* if it has no wandering sets of positive measure. If it does have wandering sets of positive measure then it is called *dissipative.*

Theorem 11 (Hopf decomposition). *There is a partition $\hat{X} = \hat{X}_c \cup \hat{X}_d$ into two disjoint measurable subsets which are characterized uniquely modulo null sets by the following properties:*

1. *$\hat{X}_c$ is invariant by $\hat{f}$ and the restriction of $\hat{f}$ to $\hat{X}_c$ is conservative.*
2. *$\hat{X}_d$ is a countable union of wandering sets.*

The sets $\hat{X}_c$ and $\hat{X}_d$ are called the conservative *and* dissipative *part of $\hat{f}$, respectively.*

The conservative part $\hat{X}_c$ can be split further into two parts $\hat{X}_+$ and $\hat{X}_0 := \hat{X}_c \setminus \hat{X}_+$, where $\hat{X}_+$ is the largest set supporting a finite absolutely continuous invariant measure. $\hat{X}_+$ is called the called the *positively recurrent* part and $\hat{X}_0$ is called the *null recurrent* part of $\hat{P}$.

Definition 12. For a nonnegative measurable function $\hat{\psi}$ let

$$\hat{\mathrm{S}}_n \, \hat{\psi} := \sum_{k=0}^{n-1} \hat{\mathrm{P}}^k \, \hat{\psi} \quad \text{and} \quad \hat{\mathrm{A}}_n \, \hat{\psi} := \frac{1}{n} \sum_{k=0}^{n-1} \hat{\mathrm{P}}^k \, \hat{\psi}$$

and denote by $\hat{\mathrm{S}}_\infty \, \hat{\psi}$ and $\hat{\mathrm{A}}_\infty \, \hat{\psi}$ the corresponding limits for $n \to \infty$ if they exist.

Roughly speaking, for a given initial distribution $\hat{\psi}$ the sum $\hat{\mathrm{S}}_n \, \hat{\psi}(\hat{x})$ measures the total amount of mass that passes through the point $\hat{x}$ in n time steps when the distribution evolves under the influence of $\hat{f}$, and $\hat{\mathrm{A}}_n \, \hat{\psi}(\hat{x})$ measures the average amount. Consequently, the behaviour of $\hat{\mathrm{S}}_n \, \hat{\psi}$ for $n \to \infty$ can be used to distinguish the conservative and dissipative part, and the behaviour of $\hat{\mathrm{A}}_n \, \hat{\psi}$ for $n \to \infty$ can be used to distinguish the null recurrent from the positively recurrent part.

Proposition 13. *The Hopf decomposition is determined uniquely modulo null sets by the following property: If $\hat{\psi}$ is a positive integrable function then $\hat{X}_c = \{\hat{\mathrm{S}}_\infty \, \hat{\psi} = \infty\}$ a.s. and $\hat{X}_d = \{\hat{\mathrm{S}}_\infty \, \hat{\psi} < \infty\}$ a.s.*

If one starts with an integrable regular density $\hat{\psi}$ then all partial sums $\hat{\mathrm{S}}_n \, \hat{\psi}$ are again integrable regular densities, so that the sequence $(\log \hat{\mathrm{S}}_n \, \hat{\psi})_{n \in \mathbb{N}}$ is equicontinuous on compact subsets of $\hat{X}$ by Proposition 10. Therefore, the asymptotic behaviour of the sequence $(\hat{\mathrm{S}}_n \, \hat{\psi}(\hat{x}_0))_{n \in \mathbb{N}}$ at a particular point $\hat{x}_0$

should determine the behaviour of the sequences $(\hat{\mathrm{S}}_n \hat{\psi}(\hat{x}))_{n\in\mathbb{N}}$ for all other points $\hat{x}$ that belong to the same irreducible component as $\hat{x}_0$. This is the core of the following theorem, which we adopt from Keller [21] to our setting.

Theorem 14. *Fix a density $\hat{\psi} \in \mathcal{H} \cap \mathrm{L}^1_{\hat{m}}$ and a point $\hat{x}_0$ from a level $\hat{D}$ in the Hofbauer tower, and let $s_n := \hat{\mathrm{S}}_n \hat{\psi}(\hat{x}_0)$ for $n \in \mathbb{N}_\infty$. Then $s_n = O(n)$ and if $s_\infty > 0$ then the following holds.*

1. *If $(s_n)_{n\in\mathbb{N}}$ is bounded then $\hat{f}$ is dissipative on $\hat{D}$ and on every level $\hat{C} \twoheadrightarrow \hat{D}$. The sequence $(\hat{\mathrm{S}}_n \hat{\psi})_{\hat{C}}$ converges uniformly on compact sets to $(\hat{\mathrm{S}}_\infty \hat{\psi})_{\hat{C}} \in \mathcal{H}$. If $(s_n)_{n\in\mathbb{N}}$ is unbounded then $\hat{D}$ belongs to the irreducible component $\hat{\mathcal{X}}_{\mathrm{m}}$ which is maximal with respect to the ordering introduced in Theorem 6*(6) *and $\hat{f}$ is conservative on $\hat{X}_{\mathrm{m}} := \bigcup\{\hat{C} \mid \hat{C} \in \hat{\mathcal{X}}_{\mathrm{m}}\}$.*
2. *If $\hat{f}$ is conservative on $\hat{X}_{\mathrm{m}}$ then it is $\hat{m}$-ergodic and the sequence $\frac{1}{s_n} \hat{\mathrm{S}}_n \hat{\psi}$ converges uniformly on compact sets to a function $\hat{h} \in \mathcal{H}$ which is strictly positive on $\hat{X}_{\mathrm{m}}$ and vanishes everywhere else. The function $\hat{h}$ is the unique*(7) *positive measurable function which is invariant by $\hat{f}$. For every $\hat{\phi} \in \mathrm{L}^1_{\hat{m}}$*
$$\lim_{n\to\infty} \frac{1}{s_n} \hat{\mathrm{S}}_n \hat{\phi} = \frac{\int \hat{\phi}\, d\hat{m}}{\int \hat{\psi}\, d\hat{m}} \cdot \hat{h} \quad \text{holds } \hat{m}\text{-a.e. on } \hat{X}_{\mathrm{m}}.$$
3. *If $\hat{f}$ is conservative on $\hat{X}_{\mathrm{m}}$ then $s_n/n \to \gamma$ for some constant $\gamma \geqslant 0$ and $\hat{h}$ is integrable if and only if $\gamma > 0$. In particular, for every $\hat{\phi} \in \mathrm{L}^1_{\hat{m}}$*
$$\lim_{n\to\infty} \frac{1}{n} \hat{\mathrm{S}}_n \hat{\phi} = \frac{\int \hat{\phi}\, d\hat{m}}{\int \hat{h}\, d\hat{m}} \cdot \hat{h} \quad \text{holds } \hat{m}\text{-a.e. on } \hat{X}_{\mathrm{m}}.$$
where the right hand side is zero if $\int \hat{h}\, d\hat{m} = \infty$.
   (a) *If $\int \hat{h}\, d\hat{m} < \infty$ then the measure preserving system $(\hat{f}, \hat{h} \cdot \hat{m})$ is the product of an exact system with a finite rotation.*
   (b) *If $\int \hat{h}\, d\hat{m} = \infty$ then $\hat{\mathrm{P}}^n \hat{\psi}$ tends to zero uniformly on compact sets for every $\hat{\psi} \in \mathcal{H} \cap \mathrm{L}^1_{\hat{m}}$ with $\hat{\psi} \leqslant \hat{h}$.*

If $\hat{f}$ is conservative on $\hat{X}_{\mathrm{m}}$ then $\hat{f}$ is called *essentially conservative* and if this is not the case then it is called *purely dissipative*.

Asymptotic behaviour of orbits on the tower

The Hopf decomposition already provides a lot of information about the asymptotic behaviour of typical points on the tower. For example, if $\hat{f}$ is

(6) I.e., there is only a finite chain in Theorem 6 ending with $\hat{\mathcal{T}}_{m+1} = \emptyset$.

(7) In this context "unique" always means "unique up to sets of measure zero and multiplication by a constant".

essentially conservative then the ω-limit set equals $\hat{X}_{\mathrm{m}}$ for $\hat{m}$-a.e. $\hat{x}$ and if $\hat{f}$ is purely dissipative then for $\hat{m}$-a.e. $\hat{x}$ the distance $r_n(\hat{x})$ of the n^{th} iterate $\hat{x}_n$ to the endpoints of the current level $\hat{D}[\hat{x}_n]$ tends to zero for $n \to \infty$.

Theorem 15. *Let f be a Lorenz map with negative Schwarzian derivative and let $(\hat{X}, \hat{f})$ be its canonical Markov extension. Then the following holds.*

1. *If $\hat{f}$ is purely dissipative then $r_n(\hat{x}) \to 0$ for $\hat{m}$-a.e. $\hat{x} \in \hat{X}$.*
2. *If $\hat{f}$ is essentially conservative then $\omega(\hat{x}) = \hat{X}_{\mathrm{m}}$ for $\hat{m}$-a.e. $\hat{x} \in \hat{X}$. Moreover,*
   (a) *if $\int \hat{h}\, d\hat{m} = 1$ then $\frac{1}{n}\sum_{k=0}^{n-1} \delta_{\hat{x}_k} \to \hat{\mu}$ weakly as $n \to \infty$ for $\hat{m}$-a.e. $\hat{x}$.*
   (b) *if $\int \hat{h}\, d\hat{m} = \infty$ then $\lim_{n\to\infty} \frac{1}{n} \operatorname{card}\{k < n \mid r_k(\hat{x}) \geqslant \varepsilon\} = 0$ holds for $\hat{m}$-a.e. $\hat{x}$ and every $\varepsilon > 0$.*

The global attractor

The information about the asymptotic behaviour on the tower can be projected down to the original system and one obtains the following information about the global attractor of a Lorenz map f with negative Schwarzian derivative.

Theorem 16. *If f is a Lorenz map with negative Schwarzian derivative then f has a unique global attractor A which is the union of one or two minimal attractors and for m-almost every point x the ω-limit set* coincides *with a minimal attractor. More precisely, one of the following three cases applies.*

1. *If f has an attracting periodic orbit then A is the union of one or two attracting periodic orbits, each of which attracts at least one of the critical values c_1^+ and c_1^-.*

Now assume that f has no attracting periodic orbits. Then $\omega(x) = A$ for m-a.e. x and one of the following holds.

2. *If f is infinitely often renormalizable then $A = \omega(c_1^+) = \omega(c_1^-)$. If f is infinitely often properly renormalizable then A is a Cantor set.*
3. *If f is finitely often renormalizable then either*
   (a) *A is a finite union of intervals*
   or A is a nowhere dense set which is of one of the following types.
   (b) $A = \omega(c_1^+) = \omega(c_1^-)$.
   (c) $A = \omega(c_1^\pm) \supset \mathrm{cl}(\mathrm{orb}(c_1^\mp))$ *with* $c_1^\pm \in \omega(c_1^\pm)$.
   (d) $A = \mathrm{cl}(\mathrm{orb}(c_1^\pm)) \supset \mathrm{cl}(\mathrm{orb}(c_1^\mp))$ *with* $c_1^\pm \notin \omega(c_1^\pm)$.

The only possibility where two attractors can coexist is the case of two attracting periodic orbits.

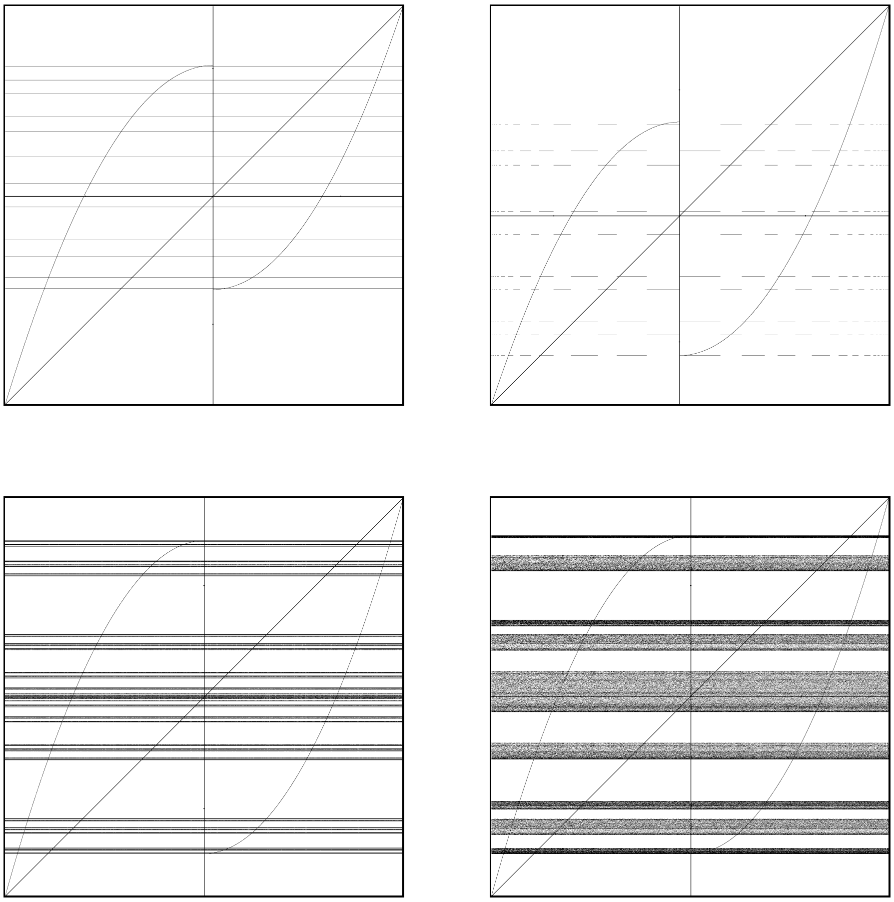

Figure3. Lorenz maps with one periodic attractor, two periodic attractors, an infinitely often renormalizable Cantor attractor, and an interval attractor (from left to right).

Figure 3 shows examples of Lorenz maps with attractors of types 1–3a. The attractor is visualized by plotting an approximation of $\omega(x)$ as function of x together with the map.

This theorem is very similar to the well known metric classification of attractors for unimodal maps by Blokh and Lyubich (cf. [5, Theorem V.1.3.]) except for cases 3c and 3d. In fact, there is a very close relationship between symmetric Lorenz maps and unimodal maps: A Lorenz map $f : [p, q] \to [p, q]$ is called *τ-symmetric* if there exists a continuous involution $\tau \neq \mathrm{id}$ with $\tau(c) = c$ and $f \circ \tau = \tau \circ f$. It is easily seen that a Lorenz map f is τ-symmetric iff there exists a unimodal map $g : [p, q] \to [p, q]$ with $g \circ \tau = g$ such that $f(x) = g(x)$ for $x < c$ and $f(x) = \tau \circ g(x)$ for $x > c$. Now if g has an attractor of one of the types 1–3b from [5, Theorem V.1.3.] and if the involution τ is absolutely continuous then the symmetric Lorenz map f has an attractor of the same type.(8)

This immediately shows that all attractors 1–3a actually occur in the family $f_a : x \mapsto (-a + x^2) \cdot \mathrm{sgn}(x)$ of symmetric quadratic Lorenz maps and attractors of type 3b occur in the family $f_a : x \mapsto (-a + |x|^\alpha) \cdot \mathrm{sgn}(x)$ if $\alpha > 1$ is large enough, see Bruin et al. [3]. Also, it is not hard to construct asymmetric examples for types 1–3b which are renormalizable to one of the symmetric examples above.

Attractors of types 3c and 3d have no unimodal counterpart and it is not clear whether they actually occur. At least they are possible from a topological viewpoint, since we are able to construct examples of Lorenz maps for which the orbits of c^+ and c^- meet the required properties.

Remark 17. Attractors of type 3d appear to be a little bit exotic: If a Lorenz map has an attractor of type 3d, say, if $A = \mathrm{cl}(\mathrm{orb}(c_1^+))$, then A is the disjoint union of two sets A_α and A_ω, where A_α is a finite or countably infinite set of points that are isolated in A and $A_\omega = \omega(A_\alpha)$ is a nowhere dense set which accumulates on the left hand side but not on the right hand side of c. This implies that such maps exhibit the following kind of intermittent behaviour: The orbit of a typical point x will spend most of its time near A_ω until some iterate of x happens to land very close to c on the right hand side, where it is separated from A_ω in the next step and has to start all over again following A_α, and so on.

SRB-measures

If the lifted map $\hat{f}$ has a finite absolutely continuous invariant measure $\hat{\mu}$ then the measure can be projected from the tower to the base and one obtains an absolutely continuous ergodic invariant measure μ of positive entropy for f.

(8) Observe that an orientation *reversing* n-periodic attractor for g yields one symmetric $2n$-periodic attractor for f and an orientation *preserving* n-periodic attractor for g yields two distinct n-periodic attractors for f which are mirror images of each other.

Theorem 18. *If f is a Lorenz map with negative Schwarzian derivative then f has an absolutely continuous invariant measure μ of positive entropy if and only if $\hat{f}$ is positively recurrent on $\hat{X}_{\mathrm{m}}$. If this is the case then $\mu = h \cdot m$, where $h := \mathrm{P}_\pi \hat{h}$, μ is ergodic and $\lambda^+(x) > 0$ on a set of full Lebesgue measure.*

It was already said that such a measure is an SRB-measure. If this is not the case then still the following can be said about the weak limit sets of the empirical distributions for typical points.

Theorem 19. *If f is a Lorenz map with negative Schwarzian derivative which has no absolutely continuous invariant probability measure of positive entropy then $\omega^*(x)$ is contained in the convex closure of $\omega^*(c_1^+) \cup \omega^*(c_1^-)$ and $\lambda^+(x) \leq 0$ for a.e. $x \in X$.*

The reason for this is that the orbit of a typical point $\hat{x}$ spends more and more of its time climbing very high up in the tower, which implies that in average there are longer and longer blocks $x_{m+1}, \dots, x_{m+n}$ where the orbit of x shadows one of the critical orbits $c^\pm$ in the sense that x_{m+i} and $c_i^\pm$ lie on the same side of the critical point for $i = 1, \dots, n$. Although this does not mean that the distance $\mathrm{dist}(x_{m+i}, c_i^\pm)$ is small during the entire shadow—even more since one cannot exclude the existence of wandering intervals for Lorenz maps in general—it is possible to show that at least it is small most of the time when the length of the shadow is large enough. The dichotomy in terms of $\lambda^+(x)$ expressed by Theorems 18 and 19 is taken from [21].

4 Kneading theory

The kneading invariant

The most important property that distinguishes one-dimensional dynamical systems from higher-dimensional systems is the presence of an order structure on $\mathbb{R}$. This order structure is heavily exploited by kneading theory: Since the order between two points x and y is preserved by a Lorenz map under iteration as long as the iterates x_n and y_n lie on the same side of the discontinuity, the information about the relative position of the iterates of a point x with respect to the discontinuity is of great importance. This information can be encoded in a binary sequence $\zeta(x) := (\zeta_n(x))_{n \geqslant 0}$, where $\zeta_k(x) := 0$ if $f^k(x) < c$ and $\zeta_k(x) := 1$ if $f^k(x) > c$, which is called the *itinerary* of x. Obviously, the map ζ is monotonous from the interval to the one-sided shift space and satisfies $\zeta \circ f = \sigma \circ \zeta$, where σ denotes the shift map. If one modifies the interval by doubling all preiterates of the critical point topologically and extends ζ to the resulting topological space X_f by taking one-sided limits then ζ is a semiconjugacy onto a closed invariant subshift Σ_f of the one-sided shift space. It is a conjugacy if and only if f is topologically expansive.(9)

(9) A Lorenz map f is called *topologically expansive* if there exists some $\varepsilon > 0$ with the following property: For any two distinct points x and y there exists some n such that $d(x_n, y_n) \geqslant \varepsilon$.

The most important itineraries are the two limits at the critical point, together forming the *kneading invariant* $\nu := (\nu^+, \nu^-) := (\zeta(c^+), \zeta(c^-))$ of the Lorenz map f. The kneading invariant completely determines the set of all admissible itineraries of points in the dynamical interval: A sequence $\omega \in \Sigma$ occurs as the itinerary of some point $x \in X_f$ if and only if it satisfies

$$\sigma\nu^+ \leqslant \sigma^n\omega \leqslant \sigma\nu^- \qquad \forall\, n \geqslant 0 \;.$$

Since the above equation applies in particular to the itineraries ν^+ and ν^-, any pair of binary sequences that occurs as the kneading invariant of some Lorenz map must necessarily satisfy the following *admissibility condition.*

Definition 20 (admissibility condition). A pair $\nu = (\nu^+, \nu^-)$ of binary sequences is called *admissible*, if

$$\sigma\nu^+ \leqslant \sigma^n\nu^\pm \leqslant \sigma\nu^- \quad \text{for every } n \in \mathbb{N}. \tag{AC}$$

Here and in the following we tacitly assume that $\nu = (10{*}, 01{*})$.

Conversely, given an admissible pair we are able to construct a continuous Lorenz map with precisely this pair as kneading invariant. This yields the following theorem.

Theorem 21. *A pair $\nu = (\nu^+, \nu^-)$ of binary sequences is realizable as the kneading invariant of a Lorenz map f if and only if it satisfies condition* (AC).

This theorem completes a theorem of Hubbard & Sparrow [18] where they showed that a pair ν of binary sequences is realizable as the kneading invariant of a topologically expansive Lorenz map f if and only if it satisfies the following *expansive admissibility condition:*

$$\sigma\nu^+ \leqslant \sigma^n\nu^+ < \sigma\nu^- \quad \text{and} \quad \sigma\nu^+ < \sigma^n\nu^- \leqslant \sigma\nu^- \quad \text{for every } n \in \mathbb{N}. \tag{EAC}$$

Splitting itineraries and admissibility conditions

The kneading invariant and the Markov diagram are two equivalent methods to describe the combinatorial behaviour of a Lorenz map in the sense that it is possible to translate the combinatorial information contained in the kneading invariant into the language of the Markov diagram, and vice versa. The key to this equivalence is the fact that the cutting times of the Hofbauer tower and hence the kneading map can be determined through a simple splitting technique from the kneading invariant: The first cutting$^+$ time is $S_0^+ = 1$ and if $m = S_{k-1}^+$ is a cutting$^+$ time, then $S_k^+ = n$ where $n > m$ is the minimal integer such that $c_n^+ < c < c_{n-m}^-$, i.e, the minimal integer at which the itineraries of c_{m+1}^+ and c_1^- differ:

$$\underbrace{\nu_{m+1}^+ \dots \nu_{n-1}^+}\nu_n^+ = \underbrace{\nu_1^- \dots \nu_{n-m-1}^-}\mathring{\nu}_{n-m}^- \;,$$

where $\mathring{\nu}_k^- := 0$ if $\nu_k^- = 1$, and vice versa. Continuing in this way one obtains a splitting of ν^+ into blocks $\Delta_k^- := \nu^+_{S^+_{k-1}+1}\nu^+_{S^+_{k-1}+2}\dots\nu^+_{S^+_k}$, i.e.,

$$\nu^+ = 1\,0\,\Delta_1^-\,\Delta_2^-\,\Delta_3^-\,\dots\ ,$$

where each block Δ_k^- is a copy of the itinerary $\nu_1^-\nu_2^-\dots$ except for the last digit. (It may happen that there are only finitely many blocks and the last block has infinite length.) The cutting$^-$ times can be determined symmetrically.

The splitting technique was already used by Hofbauer and then systematized by Bruin [1,2] and Sands [32]. They introduced the so-called co-cutting times which provide additional information about the recurrence behaviour of the critical point. For Lorenz maps, the co-cutting$^+$ times can be obtained by the following modified splitting of ν^+: Starting from the smallest index greater than $S_0^+ = 1$ which is no cutting$^+$ time (i.e., the second digit that equals 1) one splits the itinerary ν^+ into blocks $\widetilde{\Delta}_k^+ := \nu^+_{\tilde{S}^+_{k-1}+1}\nu^+_{\tilde{S}^+_{k-1}+2}\dots\nu^+_{\tilde{S}^+_k}$, i.e.,

$$\nu^+ = 1\,0^{\tilde{S}_0^+-1}1\,\widetilde{\Delta}_1^+\,\widetilde{\Delta}_2^+\,\widetilde{\Delta}_3^+\,\dots\ ,$$

where each block $\widetilde{\Delta}_k^+$ is a copy of the itinerary $\nu_1^+\nu_2^+\dots$ except for the last digit. The co-cutting$^-$ times are defined symmetrically.

Note that the splitting of a kneading is obtained by comparing w.r.t. the other kneading, whereas the co-splitting is obtained by comparing w.r.t. itself. Using this technique we obtain a reformulation of the *admissibility condition* for kneading invariants in the language of the Markov diagrams, i.e., a necessary and sufficient combinatorial condition for an abstract graph to occur as the Markov diagram of some Lorenz map.

Theorem 22. *For an arbitrary pair* $(\nu^+,\nu^-) = (10*, 01*)$ *of binary sequences the following conditions are equivalent*

1. *For every* $n \in \mathbb{N}$ *the inequality* $\sigma\nu^+ \leqslant \sigma^n\nu^\pm \leqslant \sigma\nu^-$ *holds.*
2. *A cutting*$^\pm$ *is never a co-cutting*$^\pm$ *time and vice versa.*
3. *The differences between consecutive cutting*$^\pm$ *times are cutting*$^\mp$ *times and the differences between consecutive co-cutting*$^\pm$ *times are cutting*$^\pm$ *times.*
4. *The differences between consecutive cutting*$^\pm$ *times are cutting*$^\mp$ *times, i.e., there exist integers* $Q^\pm(k)$, $k \geqslant 1$, *such that*

$$S_k^\pm - S_{k-1}^\pm = S^\mp_{Q^\pm(k)}$$

and the kneading map $Q^\pm$ *satisfies the* Hofbauer condition*:*

$$\left(Q^\pm(k+j)\right)_{j\geqslant 1} \geqslant \left(Q^\pm(Q^\mp Q^\pm(k)+j)\right)_{j\geqslant 1} \quad \forall k \geqslant 1\ ,$$

where the ordering $\leqslant$ *is just the lexicographical ordering of sequences.*

Although at first sight the formulation of admissibility in terms of the kneading map (condition 4) looks much more complicated than the other three, it is in practice very powerful, since it contains the essential information about the topological properties of the Lorenz map in a condensed form.[(10)] This makes it very easy to construct Lorenz maps with prescribed topological properties, e.g., renormalizable maps or maps where the critical points are recurrent, respectively not recurrent, and so on.

5 Families of Lorenz maps

Since the original motivation for studying Lorenz maps came from the analysis of homoclinic bifurcations of vector fields, it is only natural to study parametrized families of Lorenz maps and to ask how the dynamics of the maps in this family depends on its parameters.

Full families

Given a family of Lorenz maps, we ask for sufficient conditions which guarantee that the maps of this family exhibit all different topological types of dynamical behaviour in the sense that every admissible kneading invariant occurs for a map in this family.

Definition 23. A family $\mathcal{F}$ of Lorenz maps is called a *full family* if every admissible kneading invariant ν occurs as the kneading invariant of some map $f \in \mathcal{F}$.

An obvious requirement for such a family is that should at least contain maps with "all different kinds of branches", i.e., by adjusting the parameters it should be possible to tune the length of both branches independently over the whole range—from short (i.e., not critical) to long (i.e., surjective) branches. This is the case for the following two parameter family of Lorenz maps.

Definition 24 ($\mathcal{C}^1$-Lorenz family). Let Λ be an open simply connected subset of $\mathbb{R}^2$. A family $\mathcal{F} := (f_{a,b})_{(a,b)\in\Lambda}$ of maps $f_{a,b} : \mathbb{R} \to \mathbb{R}$ is called a $\mathcal{C}^1$*-Lorenz family* if the following properties hold.

1. The map $(a,b,x) \mapsto f_{a,b}(x)$ is continuously differentiable for $x \neq c := 0$ and the one-sided limits $\mathrm{D}_a\, f_{a,b}(0^\pm) = \mathrm{D}_a\, c_1^\pm(a,b)$ and $\mathrm{D}_b\, f_{a,b}(0^\pm) = \mathrm{D}_b\, c_1^\pm(a,b)$ exist for all parameters $(a,b) \in \Lambda$.
2. $\mathrm{D}_x\, f_{a,b}(x) > 0$ and $\mathrm{D}_x\, f_{a,b}(x) \to 0$ for $x \to 0$.

(10) The dynamical significance of the Hofbauer condition is not so easily unveiled. We only remark here that it is trivially satisfied if both sequences Q^+ and Q^- are monotonously increasing.

3. The map $f_{a,b}$ has precisely two fixed points $p^-(a,b) < 0$ and $p^+(a,b) > 0$ which are hyperbolic repellers.
4. There are no neutral or repelling fixed points in $(p^-(a,b), p^+(a,b))$.

Additionally, we assume that $\mathbb{R}^2_+ \subseteq \Lambda$ and that the following properties hold.

5. $c_1^+(0,b) = 0$ and $c_1^-(a,0) = 0$ for all $a,b \geqslant 0$.
6. $\mathrm{D}_a\, c_1^+(a,b) < 0$ and $\mathrm{D}_b\, c_1^-(a,b) > 0$
7. $\mathrm{D}_a\, f_{a,b}(x) \leqslant 0$ and $\mathrm{D}_b\, f_{a,b}(x) \geqslant 0$ for $x \in (p^-(a,b), p^+(a,b))$.
8. There exists a parameter such that $f_{a,b}$ has full branches, i.e., $c_1^+(a,b) = p^-(a,b)$ and $c_1^-(a,b) = p^+(a,b)$.

The first four conditions ensure that the family depends $\mathcal{C}^1$-smoothly on its parameters and is reasonably well behaved. The last four conditions guarantee that the branches can be adjusted arbitrarily and that the branches move monotonously as long as only one of the two parameters is modified.

Prototypes of such $\mathcal{C}^1$-Lorenz families are the families

$$f_{a,b} : x \mapsto \begin{cases} -a + |x|^\alpha & , \text{if } x > 0 \\ \ \ b - |x|^\alpha & , \text{if } x < 0 \end{cases} \tag{1}$$

for fixed constant $\alpha > 1$, in particular the *quadratic Lorenz family*

$$f_{a,b} : x \mapsto \begin{cases} -a + x^2 & , \text{if } x > 0 \\ \ \ b - x^2 & , \text{if } x < 0 \end{cases} . \tag{2}$$

The parameter space Λ is chosen unnecessarily large and not every map of such a $\mathcal{C}^1$-Lorenz family is really a Lorenz map in the sense that it has an invariant interval restricted to which it looks like the map in Figure 1. However, under the assumptions of the definition it can be shown that there is a bounded simply connected open region $J \subset \Lambda$ such that $p^- < c^+ < 0 < c^- < p^+$ holds for every map $f_{a,b}$, $(a,b) \in J$ and that the boundary of J is the union of four curves where the maps satisfy $c^+ = 0$, $c^+ = p^-$, $c^- = 0$, and $c^- = p^+$, respectively.

Now the answer to the above question is: Every $\mathcal{C}^1$-Lorenz family is a full family—well, almost. There is an exceptional set of kneading invariants which do not necessarily occur, namely the ones where one of the critical itineraries, is a shift of the other one, say, $\nu^+ = \sigma^k \nu^-$ for some $k > 0$,[11] but ν^+ is not periodic. Whenever a Lorenz map has such a kneading invariant, a one-sided neighbourhood of the critical point is necessarily a wandering interval or contained in the basin of an inessential periodic attractor. Such kneading invariants may or may not occur in a $\mathcal{C}^1$-Lorenz family. For families of maps with negative Schwarzian derivative like the quadratic Lorenz family they are certainly missing (cf. [33]). The following theorem shows that this

[11] This implies that ν is not expansive, i.e., condition (EAC) on page 17 is violated.

is the only exception. We call it the *Full Family Theorem*, regardless of its little imperfections.

Theorem 25. *Let $(f_{a,b})_{(a,b)\in J}$ be a C^1-Lorenz family and let $\nu = (\nu^+, \nu^-)$ be an admissible kneading invariant such that*

i) ν is expansive or ii) at least one of ν^+ and ν^- is periodic.

Then there exists a parameter $(a,b) \in \operatorname{cl} J$ where $f_{a,b}$ has the kneading invariant ν.

We sketch a proof of this theorem which is an adaptation of the proof of the Full Family Theorem for smooth multimodal maps by de Melo & van Strien [5] (see also Martens & de Melo [26]). It is based on the so-called Thurston map which provides an elegant method for solving the following finite version of the problem posed above: Given a kneading invariant as in the theorem with the additional property that both itineraries ν^+ and ν^- are preperiodic, find a parameter where the Lorenz map is *post-critically finite*(12) and has the required kneading invariant. This problem can be reformulated as a fixed point problem for the Thurston map which is a continuous map from a finite dimensional simplex into itself. Since the fixed point is required to lie in the interior of the simplex, the Brouwer Theorem is not applicable immediately, but a closer look at its proof shows that in the given situation it can be modified to prove the existence of such a fixed point.

Although the analysis of the parameter dependence of the kneading invariant yields an independent proof of Theorem 25 as a byproduct, we included the discussion of the Thurston map approach, not only because of its great theoretical importance, but also for a very practical reason: The Thurston algorithm provides a convenient method to find Lorenz maps with specific combinatorial properties in the quadratic Lorenz family which can easily be implemented on a computer. To be honest, one has to cheat a little bit. The algorithm is based on the assumption that the Thurston map for the quadratic family (2) is a contraction with respect to some suitably chosen metric and the fixed point is found by iterating an arbitrary initial value. Unfortunately, we are not able to show that this is indeed the case, but the fact that the Thurston algorithm works very nicely and reliable in practice gives strong evidence for the following conjecture.

Conjecture 26. The Thurston map for the quadratic Lorenz family is a contraction with respect to a suitable chosen metric.

This conjecture implies that for every preperiodic admissible kneading invariant as in Theorem 25 there is precisely one parameter (a,b) such that the Lorenz map $f_{a,b}$ is post-critically finite and its kneading invariant coincides

(12) This means that the set $\operatorname{orb}(c^+) \cup \operatorname{orb}(c^-)$ is finite, i.e., both critical points are preperiodic.

with the given one. This uniqueness problem is closely related to the question of monotonous dependence of the kneading invariant for the quadratic family which we are going to adress below.

The bifurcation diagram

Of course it is even more interesting not only to know whether all admissible sequence actually occur but also where in parameter space they can be found and how the kneading invariant changes with the parameters. In order to do this we describe the bifurcation diagram of a $\mathcal{C}^1$-Lorenz family, which consists of a refining sequence of partitions of its parameter space that is obtained by distinguishing longer and longer initial parts of the kneading invariants.

Definition 27. Let $(f_{a,b})_{(a,b)\in J}$ be a $\mathcal{C}^1$-Lorenz family. For every $n \in \mathbb{N}$ and every admissible kneading invariant $\nu = (\nu^+, \nu^-)$ let $J_n(\nu)$ be the set of parameters (a, b) such that

1. the critical points $c^+(a, b)$ and $c^-(a, b)$ of the map $f_{a,b}$ are not periodic of any period less than or equal to n, and
2. the kneading invariant $\nu(a, b)$ of $f_{a,b}$ coincides up to the n^{th} digit with ν.

We say that the *admissible n-prefix* $(\nu_0^+ \dots \nu_n^+, \nu_0^- \dots \nu_n^-)$ is the *combinatorial type* of $J_n(\nu)$. The collection of partitions $\mathcal{J}_n := \{J_n(\nu) \mid \nu \text{ admissible }\}$ is called the *bifurcation diagram* of the Lorenz family $\mathcal{F}$.

The bifurcation diagram of the quadratic Lorenz family is shown in Figure 4. The region shaped like an almond is the parameter space J, which is contained in the square $[0, 2] \times [0, 2]$. Whenever the index of the partition is increased by one, some pieces of the partition break into fragments—which can be of two or four different combinatorial types—while other pieces remain unchanged. The number of possible combinatorial subtypes for a given piece is limited by the number of extensions of the given admissible $(n-1)$-prefix to an admissible n-prefix and can easily be determined using condition (2) of Theorem 22. Conversely, the following theorem shows that in a $\mathcal{C}^1$-Lorenz family every admissible subtype actually occurs.(13)

Theorem 28. *Let $(f_{a,b})_{(a,b)\in J}$ be a $\mathcal{C}^1$-Lorenz family and let $\nu = (\nu^+, \nu^-)$ be an admissible kneading invariant.*

Then there is a decreasing sequence $(G_n(\nu))_{n\in\mathbb{N}}$ of nonvoid simply connected open regions $G_n(\nu) \subseteq J_n(\nu)$. The boundary of each set $G_n(\nu)$ is contained in the union of four differentiable arcs where one of the critical points is either periodic of some period less than or equal to n (which is determined by the cutting and co-cutting times) or is mapped onto one of the repelling fixed points in the boundary.

(13) A similar argument was used by Hofbauer & Keller [16] to prove the Full Family Theorem for unimodal maps.

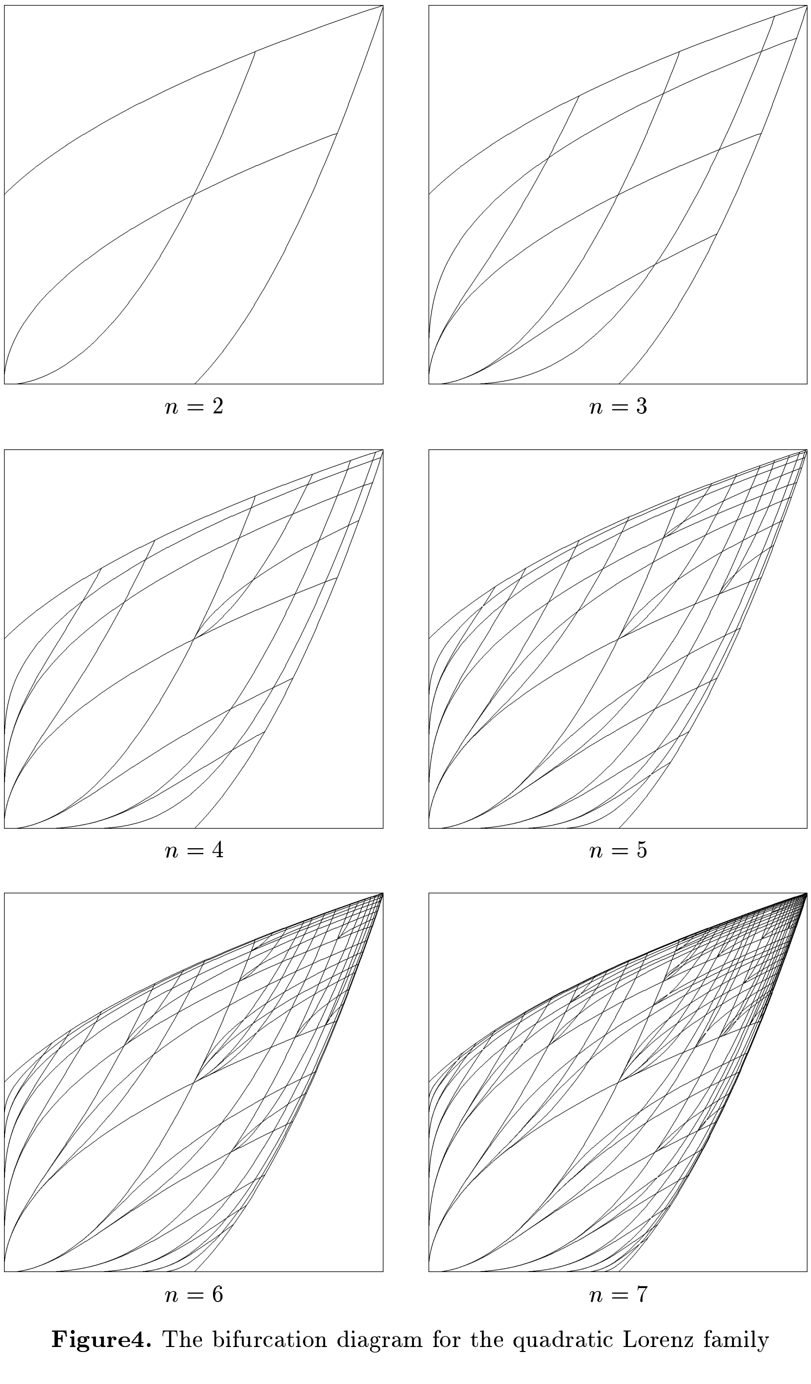

Figure4. The bifurcation diagram for the quadratic Lorenz family

For every parameter (a, b) in $G_\infty(\nu) := \bigcap_{n\in\mathbb{N}} G_n(\nu)$ the critical points are not periodic and the Lorenz map $f_{a,b}$ has the kneading invariant ν. If ν is expansive or periodic then $G_\infty(\nu)$ is nonvoid. In particular, this proves Theorem 25.

The main tool for the proof of this theorem is a local analysis of the bifurcations at the border lines between two adjacent pieces of the partition $\mathcal{J}_n$, in particular at the intersections of these lines which form the *corners* of the pieces $J_n(\nu)$. Since the kneading invariant changes there, one or both of the critical points has to be periodic of some period less than or equal to n.

Definition 29. A periodic orbit containing one of the critical points is called a *homoclinic orbit*. Given a finite binary word $w_\pm$ of length $n^\pm$, $c^\pm$ is called *homoclinic* of type $w_\pm$ if $c^\pm_{n^\pm} = 0$, $c^\pm_k \neq 0$ for $k < n^\pm$, and $\nu^\pm = w_\pm^\infty$. The set of parameters where $c^\pm$ is homoclinic of type $w_\pm$ is denoted by $C_{w_\pm}$. Connected components of $C_{w_\pm}$ are called *hom-lines*.

In Figure 4 the lines going from left to right respectively from bottom to top are hom-lines where c^- respectively c^+ is homoclinic.

Remark 30. The term "homoclinic orbit" is motivated by the origin of Lorenz maps as first return maps for flows on the geometric Lorenz attractor, where periodic critical orbits of the Lorenz map correspond to homoclinic loops on the geometric Lorenz attractor.

At the intersection points of two hom-lines C_{w_+} and C_{w_-}, which we call *hom-points* of type (w_+, w_-), the Lorenz map has two homoclinic orbits. For nearby parameters in a quadrant northeast or souththwest of the hom-point—depending on the orientation of the intersection—the maps are renormalizable to a Lorenz map which is injective on its dynamical interval. Such maps can be considered as injective circle maps, whence they have a well defined rotation number. It can be shown that there are infinitely many hom-lines emerging from the hom-point in direction of this quadrant which form the boundaries of the frequency locked regions with rational rotation number. The combinatorial types of these hom-lines can be obtained from the original two types w_+ and w_- using concatenation rules. This type of bifurcation is known as the *gluing bifurcation*, see Coullet e.a. [4] and Gambaudo e.a. [6].

We distinguish *positively* and *negatively oriented* hom-points according to the orientation of the intersection of C_{w_+} and C_{w_-}. The local picture at a hom-point is summarized by the following proposition (cf. Figure 5).

Proposition 31. *Consider a transverse hom-point (a_0, b_0) of type (w_+, w_-) and let U be a sufficiently small neighbourhood of (a_0, b_0) which is divided into four quadrants by the hom-lines C_{w_+} and C_{w_-}. If the hom-point is positively oriented then one has the following bifurcation diagram.*

1. *Southwest of* (a_0, b_0) *the map* f *has two periodic orbits of type* w_- *resp.* w_+ *containing* c^- *resp.* c^+ *in their immediate basin. The kneading invariant equals* (w_+^∞, w_-^∞).
2. *Northwest of* (a_0, b_0) *the map* f *has a periodic orbit of type* w_+ *containing* c^+ *and* $c_{n^-}^-$ *in its immediate basin. The kneading invariant equals* $(w_+^\infty, w_- w_+^\infty)$.
3. *Southeast of* (a_0, b_0) *the map* f *has a periodic orbit of type* w_- *containing* c^- *and* $c_{n^+}^+$ *in its immediate basin. The kneading invariant equals* $(w_+ w_-^\infty, w_-^\infty)$.
4. *Northeast of* (a_0, b_0) *the map* f *is* (w_+, w_-)*-renormalizable to an injective circle map. There are infinitely many hom-lines emerging from* (a_0, b_0) *into this direction corresponding to hom-lines of the renormalized circle map.*

If the orientation is negative then the same holds with the roles of the northeast and the southwest quadrant interchanged.

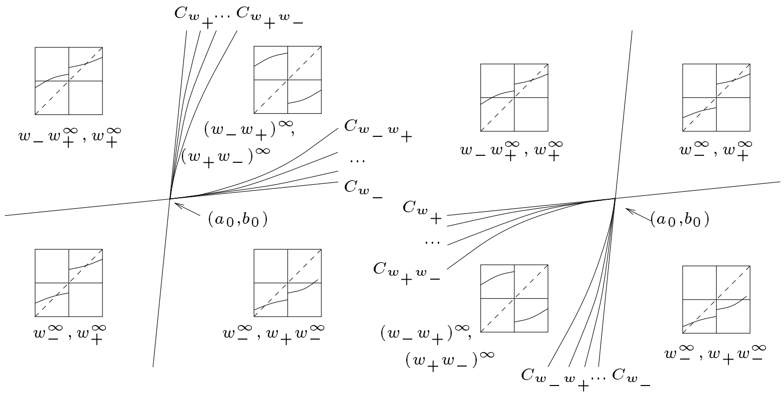

Figure5. If the hom-point is positively oriented (left figure) then northeast of the homoclinic value (a_0, b_0) the map f becomes renormalizable to an injective circle map and infinitely many new hom-lines are created. Similarly, infinitely many curves are annihilated in a negatively oriented hom-point (right figure).

Finally, the combinatorial information obtained from the kneading invariant and the local analysis at the corners of the pieces $J_{n-1}(\nu)$ are combined in order to show that for every admissible extension of the given $(n-1)$-prefix of ν there is at least one corner of $J_{n-1}(\nu)$ where this prefix can be found (cf. Figure 6). Now Theorem 28 follows by induction.

Monotonicity of the kneading invariant

If $(f_a)_{a\in\mathbb{R}}$ is a $\mathcal{C}^1$-family of *unimodal* maps then one says that the kneading invariant depends monotonously on the parameter, if the map $a \mapsto \nu(a)$ is a

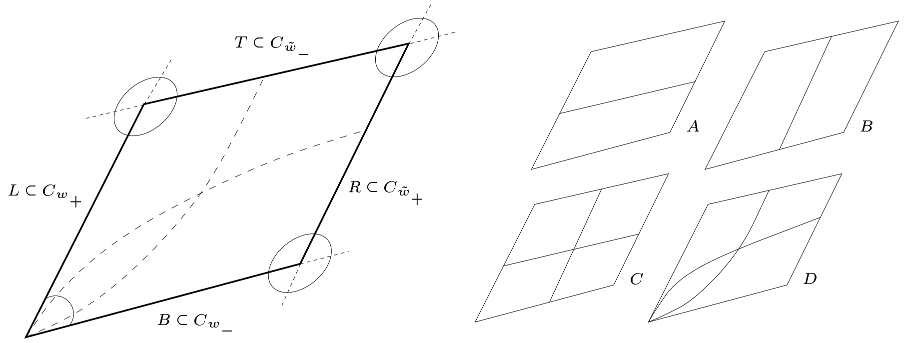

Figure6. The edges of a piece $J_{n-1}(\nu)$ are parts of hom-lines C_{w_+}, $C_{\tilde{w}_+}$, C_{w_-} and $C_{\tilde{w}_-}$ and the corners are hom-points. If $J_{n-1}(\nu) \notin \mathcal{J}_n$ then it splits into two or four fragments of $\mathcal{J}_n$ as in one of the figures A–D.

monotonous map from the real line to the shift space $\{-,+\}^{\mathbb{N}}$ endowed with the signed lexicographical order. The following definition is a natural generalization of this notion for families depending on more than one parameter.

Definition 32. Let $\mathcal{F} = (f_{a,b})_{(a,b)\in J}$ be a $\mathcal{C}^1$-Lorenz family. We say that the kneading invariant of $\mathcal{F}$ *depends monotonously on the parameters* if for every admissible kneading invariant ν and every n the set $J_n(\nu)$ is simply connected.

In the course of proving Theorem 28 it becomes evident that a sufficient condition for the sets $J_n(\nu)$ to be simply connected is the requirement that all hom-points are positively oriented.

Theorem 33. *In the situation of Theorem 28 assume that all hom-points are positively oriented. Then the kneading invariant of the Lorenz family depends monotonously on the parameters.*

The numerical bifurcation diagrams obtained for the quadratic Lorenz family (cf. Figure 4) strongly indicate that it has this property.

Conjecture 34. Every hom-point of the quadratic Lorenz family (2) is positively oriented.

For the quadratic *unimodal* family $f_a : x \mapsto a - x^2$ it was shown by Douady, Hubbard & Sullivan[14] that the kneading invariant depends monotonously on the parameter. They showed this by proving that the Thurston map is a contraction with respect to a suitable metric using deep results from Teichmüller theory. Recently, Tsujii [34,35] gave a simplified proof for

[14] Unpublished. For a proof see the monograph of Milnor & Thurston [28].

the monotonicity in the quadratic family based on the following observations. First, the question whether the kneading invariant depends monotonously on the parameters can be reduced to a local problem, namely to show that

$$\frac{\mathrm{D}_a\, c_{n+1}(a)}{(f^n)'\,(c_1(a))} > 0 \tag{3}$$

whenever the critical point is periodic ("homoclinic") of period $n+1$. The interpretation of this property is that whenever the critical point is periodic and the parameter a is increased, $c_{n+1}(a)$ moves towards the side of c corresponding to the larger kneading invariant. Second, there is a connection between the expression on the left hand side of the inequality and the Thurston map corresponding to the periodic orbit $c_0, \ldots, c_n$. If $\mathrm{D}\,T(\tilde{c})$ denotes the linearization of the Thurston map at the fixed point $\tilde{c} := (c_0, \ldots, c_n)$ then

$$\frac{\mathrm{D}_a\, c_{n+1}(a)}{(f^n)'\,(c_1(a))} = \det(\mathrm{I} - \mathrm{D}\,T(\tilde{c}))\ . \tag{4}$$

Since the characteristic polynomial $\det(\lambda\,\mathrm{I} - \mathrm{D}\,T(\tilde{c}))$ is a real polynomial in λ which diverges to $+\infty$ for $\lambda \to +\infty$, the above inequality follows if the spectrum of $\mathrm{D}\,T$ is contained strictly inside the unit disk, i.e., if the linearized Thurston map is a contraction.

For the hom-points of the quadratic Lorenz family we derive a two-dimensional analogue to equation (4), namely

$$\det \begin{pmatrix} \frac{\mathrm{D}_a\, c^+_{n+1}}{(f^n)'(c^+_1)} & \frac{\mathrm{D}_b\, c^+_{n+1}}{(f^n)'(c^+_1)} \\ \frac{\mathrm{D}_a\, c^-_{m+1}}{(f^m)'(c^-_1)} & \frac{\mathrm{D}_b\, c^-_{m+1}}{(f^m)(c^-_1)} \end{pmatrix} (a,b) = \det(\mathrm{I} - \mathrm{D}\,T(\tilde{c}))\ ,$$

which implies that a hom-point (a, b) is positively oriented if the linearization of the Thurston map at the corresponding fixed point $\tilde{c}$ is a contraction. In particular, Conjecture 26 implies Conjecture 34. Unfortunately, it was not possible to apply the arguments of Tsujii in order to prove the missing link, namely the local contraction of the Thurston map. Nevertheless, since we have numerical evidence for both conjectures independently, we believe very much that they are true.

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