Random Attractors: Robustness, Numerics and Chaotic Dynamics

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Abstract. In this article the numerical approximation of attractors and invariant measures for random dynamical systems by using a box covering algorithm is discussed. We give a condition under which the algorithm, which defines a set valued random dynamical system, possesses an attractor close to the attractor of the original system. Furthermore, a general existence theorem for attractors for set valued random dynamical systems is proved and criteria for the robustness of random attractors under perturbations of the system are given. Our numerical algorithm is applied to the stochastically forced Duffing oscillator, which supports for certain parameter values a non–trivial random SRB measure.

1 Introduction

The qualitative behavior of an autonomous dynamical system given by the iterations of a map or the solution of an ordinary differential equation (ODE) is often characterized by invariant objects such as (global) attractors or invariant (probability) measures. Invariant measures describe the statistical behavior of a dynamical system. An attractor is a compact invariant set, which carries the asymptotic dynamics in the sense that it is approached by all trajectories under time evolution. If a global attractor exists, then it supports all invariant measures.

These "global objects" are particularly valuable in the investigation of systems with complicated ("chaotic") dynamical behavior, where reliable predictions for single trajectories are only possible for bounded time intervals.

Here we are interested in dynamics influenced by probabilistic noise. A "traditional" approach is to model this by a Markov process. In this case the evolution law of the system is given by transition probabilities instead of a deterministic map. The statistical behavior of a Markov process is often described by probability measures, which are invariant under the transition probabilities.

However, we will work in the framework of random dynamical systems (for a systematic presentation of this theory see [1]). A discrete time random dynamical system is just given by the iteration of random mappings. Stochastic differential equations (SDE's) and random differential equations (which are ODE's with a randomly varying parameter) are generators of continuous time random dynamical systems.

Iterations of iid mappings and SDE's also define Markov processes. There is a notion of invariant measures for random dynamical systems, which are closely related to the invariant measures of the corresponding Markov process. However, typically a random dynamical system has more invariant measures than just these "Markov measures". The theory of random dynamical systems provides tools which allow a more detailed analysis of the dynamics than it is possible in the framework of Markov processes. In some sense it is possible to separate "noise driven dynamics" from "deterministic" dynamics.

There is a notion of a "pathwise" defined random attractor. That is, the attractor is a (compact) set valued random variable defined on the probability space which models the noise, i.e. for (almost) every realization of the stochastic process which models the noise there is a compact set. The distance between a trajectory with random initial condition and the attractor converges to zero in probability if time tends to infinity.

Such an attractor moves (in a stationary manner) in the phase space of the system under time evolution. This movement of the attractor can be interpreted as the "noise induced part" of the dynamics. In addition, there is some dynamics inside the attractor (which may be trivial). The qualitative structure of this dynamics inside the attractor is often the same for (almost) every noise realization. In this sense it can be viewed as the "deterministic part" of the dynamics.

To be more precise, let us look at two examples. Take some self mappings of a complete metric space, which are all uniform contractions. In every time step choose randomly one of these mappings. Then the asymptotic behavior of a trajectory is independent of the initial condition. This means, that the "deterministic part" of the dynamics is trivial. The attractor consists of one random point. However, if the mappings do not have a common fixed point, then an observer sees non–trivial dynamical behavior, which is in this case exclusively due to the noise.

The situation is different for the randomly forced Duffing oscillator, which will be discussed in more detail later on in this paper. There are parameter values, where the system has a positive Lyapunov exponent, which leads to sensitive dependence on initial conditions. Our numerical images of the attractor and of the "natural" measure supported by it suggest that the dynamics of the system is a combination of "deterministic dynamics" which is chaotic and "noise induced" movements in the phase space $\mathbb{R}^2$.

For many random dynamical systems the existence of a random attractor, which also implies the existence of invariant measures, is proved. However, in many interesting cases it seems to be quite hard to obtain analytical results on the structure of the attractor and the dynamics on it (the "deterministic part"). This means that one often has to rely on numerical simulations in order to obtain information about random attractors and invariant measures.

In this article we will discuss a numerical method for the approximation of random attractors and ("natural") invariant measures supported by them. We consider a "global" approach in order to approximate these global objects. We use a random version of the subdivision algorithm which was developed by Dellnitz and Hohmann [14,15] for the approximation of deterministic attractors. This algorithm produces a sequence of box coverings of the attractor.

The "random" subdivision algorithm is described in an article by Keller and Ochs [25]. Its application to the stochastically perturbed Duffing–van der Pol oscillator gave surprising new insight in the structure of the attractor.

A simplifying statement about the present paper would be that its content is essentially a new convergence proof for this algorithm. However, on the way to this proof we develop some tools which may be of interest for their own. After giving fundamental definitions (Sect. 2) we introduce in Sect. 3 the notion of a set valued random dynamical system, where the image of a point is a compact set instead of a single point. We define attractors for set valued random dynamical systems and give a general criterion for their existence based on the existence of an attracting set. This generalizes a often used existence theorem for an attractor of a standard point valued random dynamical system.

Using the notion of set valued random dynamical systems we give criteria for robustness of random attractors under perturbations. In Sect. 4 we apply these results to the "random" subdivision algorithm, which defines for a given random dynamical system φ a sequence of set valued random dynamical systems $\hat{\varphi}_k$ each of which possessing an attractor A_k (under an assumption on φ, see Corollary 26). We show that the intersection of all A_k is the global attractor for φ.

In addition (Sect. 4.2) we discuss a method for the numerical approximation of "natural" invariant measures of random dynamical systems. In the case of random dynamical systems generated by stochastic differential equations these "natural" measures are related to invariant measures of the corresponding Markov process.

In Sect. 5 we consider as an example the stochastically forced damped Duffing oscillator. We prove the existence of a global attractor. There are parameters where the corresponding random dynamical system has a positive Lyapunov exponent. This implies that the "natural" measure is a non–trivial random Sinai–Ruelle–Bowen measure. We have calculated numerical approximations of the attractor and of this measure. In addition we consider the case when the top Lyapunov exponent is negative. Then the support of the "natural" measure is a random one point set, but the global attractor seems to be a larger set carrying some sort of "transient" chaotic dynamics.

2 Preliminaries

2.1 Random Dynamical Systems

Definition 1. A (continuous) *random dynamical system* (RDS) consists of two ingredients:

- A measure preserving flow $\vartheta = (\vartheta_t)_{t \in \mathbb{T}}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, which serves as a model for the noise. We always assume that ϑ is invertible, i.e. $\mathbb{T} = \mathbb{R}$ or $\mathbb{Z}$.
- A measurable mapping

$$\varphi : \mathbb{T}^{(+)} \times \Omega \times X \to X, \quad (t, \omega, x) \mapsto \varphi(t, \omega)x$$

  (with $\mathbb{T}^+ = \{t \in \mathbb{T} : t \geq 0\}$), where the *state space* X is a separable metric space (with metric d), such that
  - $(t, x) \mapsto \varphi(t, \omega)x$ is continuous for fixed ω,
  - φ satisfies the *cocycle property* $\varphi(0, \omega) = \mathrm{id}_X$ and $\varphi(t + s, \omega) = \varphi(s, \vartheta_t\omega) \circ \varphi(t, \omega)$ for $t, s \in \mathbb{T}^{(+)}$ and $\omega \in \Omega$.

We say φ is a random dynamical system on X over the *metric dynamical system* (Ω, ϑ).

Remark 2. (i) The assumption that X is separable is for technical reasons.

(ii) In some definitions continuity is only required in space (i.e. $x \mapsto \varphi(t, \omega)x$ is continuous for fixed t, ω). However, continuity in time is automatically true if $\mathbb{T} = \mathbb{Z}$ and is also satisfied for continuous time random dynamical systems generated by stochastic or random differential equations (see Arnold [1, Chapter 2]).

(iii) Throughout this paper all assertions about ω are assumed to hold on a ϑ invariant set of full measure (unless otherwise stated).

(iv) The cocycle property (which reduces to the flow property if φ is independent of the noise ω) implies, that the *skew product*

$$\Theta = \Theta_\varphi : \mathbb{T}^{(+)} \times \Omega \times X \to \Omega \times X, \quad (t, \omega, x) \mapsto \Theta(t)(\omega, x) := (\vartheta_t\omega, \varphi(t, \omega)x)$$

defines a measurable (semi-)flow on the product space $\Omega \times X$.

2.2 Generators

Iteration of Random Mappings. Let X be a metric space and $\mathbb{P}_0$ a probability measure on the space $C(X, X)$ of continuous mappings from X to itself. Set $\Omega := C(X, X)^{\mathbb{Z}}$ and let ϑ be the left shift on Ω. Define $\mathbb{P} := \mathbb{P}_0^{\mathbb{Z}}$. With $\vartheta_n = \vartheta^n$, $\varphi(\omega) = \omega_0$ and

$$\varphi(n, \omega) = \varphi(\vartheta_{n-1}\omega) \circ \ldots \circ \varphi(\omega)$$

for $n \geq 1$ a random dynamical system on X over (Ω, ϑ) is defined, which models the iteration of iid mappings with distribution $\mathbb{P}_0$. More generally, a random dynamical system is defined with every ϑ invariant probability measure on Ω.

If $\mathbb{P}$ is concentrated on homeomorphisms, then $\varphi(n, \omega)$ is defined for $n < 0$ via

$$\varphi(n, \omega) = \varphi(\vartheta_{-n}\omega)^{-1} \circ \ldots \circ \varphi(\vartheta_{-1}\omega)^{-1}.$$

In the same way a discrete time random dynamical system is generated if we have an arbitrary metric dynamical system (Ω, ϑ) and a measurable mapping $\varphi : \Omega \to C(X, X)$.

Stochastic Differential Equations. Consider the Stratonovich SDE

$$dx = f(x)\, dt + \sum_{i=1}^{m} g_i(x) \circ dW_i$$

on $\mathbb{R}^d$ with initial condition $x_0 \in \mathbb{R}^d$ and functions $f, g_i : \mathbb{R}^d \to \mathbb{R}^d$ for $i = 1, \ldots, m$ and $W = (W_1, \ldots, W_m)$ an m-dimensional Wiener process. If the functions f, g_i are sufficiently regular (see Theorem 2.3.36 in Arnold [1]) this equation generates a local RDS φ over (Ω, ϑ), where $\Omega = C_0(\mathbb{R}, \mathbb{R}^m)$ is the space of continuous functions ω from $\mathbb{R}$ to $\mathbb{R}^m$ with $\omega(0) = 0$ (the path space of the Wiener process) equipped with the canonical Wiener measure, and the shift ϑ is defined by $(\vartheta_t\omega)(s) = \omega(s + t) - \omega(t)$ for all $s, t \in \mathbb{R}$ and $\omega \in \Omega$. Local means that $\varphi(t, \omega)\, x$ is only defined for $\tau^-(\omega, x) < t < \tau^+(\omega, x)$, where $-\tau^-, \tau^+ \in (0, \infty]$ are the lifetimes of solutions before possible explosion. A sufficient condition for φ being a global RDS without explosion is global Lipschitz continuity of the functions f, g_i (see Arnold [1, Theorem 2.3.32]).

Random Differential Equations. Here the generator of a continuous time random dynamical system φ is a family of ODE's with parameter ω, which can be solved "pathwise" for each ω (in contrast to the SDE case) as a deterministic non–autonomous ODE. That is, φ satisfies the integral equation

$$\varphi(t, \omega)x = x + \int_0^t f(\vartheta_s\omega, \varphi(s, \omega)x) ds,$$

where (Ω, ϑ) is any continuous time metric dynamical system and f is a function from $\Omega \times \mathbb{R}^d$ to $\mathbb{R}^d$.

We say that φ solves the *random differential equation* $\dot{x}(t) = f(\vartheta_t\omega, x(t))$. This pathwise differential equation is fulfilled if $(t, x) \mapsto f(\vartheta_t\omega, x)$ is continuous and $x \mapsto f(\omega, x)$ is Lipschitz. In this case we speak of a *classical solution*. Otherwise the differential equation is a symbolic notation for the corresponding integral equation. For general conditions on f which are needed to generate a random dynamical system see [1, Theorems 2.2.1 and 2.2.2].

2.3 Markov Processes

A (homogeneous) Markov process (see e.g. [1, Appendix A.4]) on X is given by an initial probability distribution at time 0 and a family $(P(t,x,\cdot))_{t\in\mathbf{T}^+,x\in X}$ of *transition probabilities*, i.e. $P(t,x,\cdot)$ is a probability measure on X for every $t \geq 0$, $x \in X$. For a Borel set $A \subset X$ the value $P(x,t,A)$ is the probability to end up in A at time $s+t$ if one starts in the point x at time s. The "flow property" of a Markov process is the *Chapman–Kolmogorov equation*

$$P(s+t,x,A) = \int_X P(s,y,A)P(t,x,dy).$$

If $P(t,x,\cdot)$ is a point mass in $\varphi_t x$, then a Markov process reduces to a semi–flow.

An *invariant (or stationary) measure* is a Borel probability measure ρ on X with

$$\rho(A) = \int_X P(t,x,A)d\rho(x).$$

A random dynamical system φ has *independent increments*, if $\varphi(t,\cdot)$ and $\varphi(u,\vartheta_s\cdot)$ are stochastically independent if $0 \leq t \leq s$ and $u \geq 0$. This is the case if φ describes in discrete time the iteration of iid mappings or if φ is generated by an SDE.

Every random dynamical system φ with independent increments defines a Markov process with transition probabilities

$$P(t,x,A) = \mathbb{P}(\varphi(t,\omega)x \in A).$$

2.4 Invariant Measures for Random Dynamical Systems

An *invariant measure* for the random dynamical system φ over (Ω,ϑ) is a probability measure μ on $\Omega \times \mathbb{R}^d$ with marginal $\pi_\Omega\mu = \mathbb{P}$ on Ω, which is invariant under the skew product $\Theta(t)$.

If X is a standard measurable space (which is the case if X is a Borel subset of its completion), than μ has a disintegration $d\mu(\omega,x) = d\mu_\omega(x)d\mathbb{P}(\omega)$, where $(\mu_\omega)_{\omega\in\Omega}$ is a family of probability measures on X ($\mathbb{P}$–a.s. uniquely determined by μ). The invariance condition then means $\varphi(t,\omega)\mu_\omega = \mu_{\vartheta_t\omega}$ $\mathbb{P}$–a.s.

An ergodic invariant measure for a differentiable random dynamical system (i.e. X is a smooth finite dimensional manifold and $x \mapsto \varphi(t,\omega)x$ is differentiable) allows a "local" analysis of φ based on the multiplicative ergodic theorem of Oseledets ([33], see also Arnold [1, Theorem 4.2.6]), which provides a substitute of linear algebra. Exponential expansion rates for the linearized system (Lyapunov exponents, which generalize the real parts of eigenvalues; they are independent of ω and the initial value if $\mathbb{P}$ and μ are ergodic) and corresponding random linear subspaces, where these growth rates are realized (Oseledets spaces; they generalize eigenspaces) are defined for μ almost every (ω, x).

The following result is due to Crauel [7, Proposition 2.1.(ii)]:

Proposition 3. *Let μ be an ergodic invariant measure for a differentiable random dynamical system φ. Assume that all Lyapunov exponents are strictly negative (or φ is invertible and all Lyapunov exponents are strictly positive). Furthermore suppose that the support of μ_ω is compact $\mathbb{P}$–a.s.*

Then μ_ω is $\mathbb{P}$–a.s. a convex combination of finitely many point masses (whose number is independent of ω) with equal weights.

Another result of Crauel [8] relates invariant measures for Markov processes with invariant measures for random dynamical systems.

Proposition 4. *Let φ be a random dynamical system with independent increments and ρ an invariant measure for the corresponding Markov process.*

Then μ given by $\mu_\omega = \lim_{t\to\infty} \varphi(t, \vartheta_{-t}\omega)\rho$ (where lim *is meant in the sense of weak convergence) is an invariant measure for φ. The marginal of μ on X equals ρ.*

Measures μ which come from Proposition 4 are called *Markov measures.* In general a random dynamical system (with independent increments) has also invariant measures which are not Markov measures.

2.5 Random Attractors

Definition 5. (i) A *random set* is a family $(B(\omega))_{\omega\in\Omega}$ of subsets of X, such that its *graph* $B := \{(\omega, x) : x \in B(\omega)\} \subset \Omega \times X$ is product measurable. We identify random sets with their graphs.

(ii) A random set B is *forward invariant*, if $\Theta(t)B \subset B$ (i.e. $\varphi(t,\omega)B(\omega) \subset B(\vartheta_t\omega)$) for $t > 0$. It is *invariant*, if equality holds ($\Theta(t)B = B$).

(iii) A *random compact set* (resp. *random closed set*) is a random set A such that all $A(\omega)$ are compact (resp. closed) and

$$\omega \mapsto \operatorname{dist}(x, A(\omega)) := \inf_{y\in A(\omega)} d(x, y)$$

is measurable for every $x \in X$.

Remark 6. (i) If the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is complete, then the measurability condition in the definition of a random compact (closed) set is automatically fulfilled, i.e. every random set A with $A(\omega)$ compact (closed) is a random compact (closed) set.

(ii) We allow $A(\omega)$ to be empty, i.e. $\operatorname{dist}(x, A(\omega))$ can be infinity.

(iii) Random compact sets are random variables $\Omega \to \mathcal{K}(X)$, where $\mathcal{K}(X)$ denotes the space of all compact subsets of X (including $\emptyset$) endowed with the Hausdorff metric d_H and corresponding Borel σ–algebra.

The following facts on random closed sets are well known (see e.g. Arnold [1, Chapter 1.6] or Castaing and Valadier [6]).

The union of two and the intersection of countably many random compact (closed) sets is again a random compact (closed) set. The same is true for the image $\Theta(t)A$ of a random compact set with $t \in \mathbb{T}^{(+)}$. In the case of A a random closed set one has to take the ω–wise closure of $\Theta(t)A$. For every random closed (compact) set A there exists a *selector*, i.e. a random variable $x : \Omega \to X$ with $x(\omega) \in A(\omega)$ for every ω with $A(\omega) \neq \emptyset$. Furthermore every invariant random compact set A with $\mathbb{P}(A(\omega) \neq \emptyset) = 1$ supports an invariant measure.

Random attractors are invariant random compact sets which attract trajectories of points or of compact sets. They were defined and investigated by several authors (e.g. [4,9–12,17,23,26,32,34]).

Definition 7. Let φ be a random dynamical system over (Ω, ϑ) on a metric space X and $B \subset \Omega \times X$ be a forward invariant random set.

An invariant random compact set $A \subset B$ is called a B *point attractor*, if A attracts random points in probability, i.e.

$$\lim_{t\to\infty} \mathbb{P}\Big\{\omega : \operatorname{dist}(\varphi(t,\omega)x(\omega), A(\vartheta_t\omega)) > \varepsilon\Big\} = 0$$

for every $\varepsilon > 0$ and every random variable $x : \Omega \to X$ with $x(\omega) \subset B(\omega)$.

A is called a B *set attractor*, if it attracts random compact sets in probability, i.e.

$$\lim_{t\to\infty} \mathbb{P}\big\{\omega : \operatorname{dist}(\varphi(t,\omega)C(\omega), A(\vartheta_t\omega)) > \varepsilon\big\} = 0$$

for every random compact set $C \subset B$ and every $\varepsilon > 0$.

Here $\operatorname{dist}(K, L) := \sup_{x\in K} \operatorname{dist}(x, L)$ is the Hausdorff semi–distance.

In the sequel we will restrict our attention to set attractors and call them just attractors.

Remark 8. (i) The random variables $\operatorname{dist}(\varphi(t,\cdot)C(\cdot), A(\vartheta_t\,\cdot))$ and $\operatorname{dist}(\Theta(t)C, A) = \operatorname{dist}(\varphi(t, \vartheta_{-t}\,\cdot)C(\vartheta_{-t}\,\cdot), A(\cdot))$ have the same distribution due to the ϑ invariance of $\mathbb{P}$.

Here $\operatorname{dist}(K, L)(\omega) := \operatorname{dist}(K(\omega), L(\omega))$, which is measurable for random compact sets K, L.

(ii) Definition 7 was first given in [32]. Most authors require $\lim_{t\to\infty} \operatorname{dist}(\Theta(t)C, A) = 0$ $\mathbb{P}$–a.s. in the definition of a random attractor for C from a certain collection of random compact sets.

(iii) A *global attractor* corresponds to the case $B(\omega) \equiv X$. We speak of a *local* attractor, if the *basin* $B(\omega)$ contains an open neighborhood of $A(\omega)$. In general an attractor according to our definition need not be a global or local attractor. In this case we speak of a *relative* attractor.

(iv) If A is a B attractor and μ an invariant measure with $\mu(B) = 1$, then $\mu(A) = 1$ [32, Theorem 2].

Our next result shows, that in connection with attractors it suffices to look at discrete time systems.

Theorem 9. *Let φ be a random dynamical system on X over (Ω, ϑ). Fix $T > 0$ and define a new discrete time random dynamical system ψ over (Ω, ϑ_T) by $\psi(n, \omega) := \varphi(nT, \omega)$ for $n \in \mathbb{Z}^{(+)}$.*

Let B be a random set which is forward invariant under φ (and hence under ψ).

If A is a B set attractor for ψ, then there exists a B set attractor A^* for φ with $\mathbb{P}(A^*(\omega) = A(\omega)) = 1$.

Proof. (1) We first construct a random compact set A^*, which is invariant under φ (on a ϑ invariant set of full measure) and which coincides almost surely with A.

(1.1) Define $A' := \Theta_\varphi(t)A$ with $0 < t < T$. With $\psi(\omega) := \psi(1, \omega)$ we have $\mathbb{P}$–a.s.

$$\begin{aligned} \psi(\omega)A'(\omega) &= \varphi(T,\omega)\varphi(t,\vartheta_{-t}\omega)A(\vartheta_{-t}\omega) \\ &= \varphi(t,\vartheta_{T-t}\omega)\varphi(T,\vartheta_{-t}\omega)A(\vartheta_{-t}\omega) \\ &= \varphi(t,\vartheta_{T-t}\omega)A(\vartheta_{T-t}\omega) \qquad = A'(\vartheta_T\omega), \end{aligned}$$

i.e. A' is invariant under φ. From [32, Theorem 1] (see also the proof of Theorem 17 below) it follows that $A'(\omega) \subset A(\omega)$ $\mathbb{P}$–a.s. Analogously $A''(\omega) \subset A(\omega)$ $\mathbb{P}$–a.s. with $A'' := \Theta_\varphi(T-t)A$. Thus we have for ω from a set of full measure

$$A = \Theta_\psi A = \Theta_\varphi(T)A = \Theta_\varphi(t)A'' \subset \Theta_\varphi(t)A = A' \subset A,$$

i.e. $A = A'$, which means $\mathbb{P}\{\omega : \varphi(t,\omega)A(\omega) = A(\vartheta_t\omega)\} = 1$ for every fixed $t > 0$.

(1.2) However, we want this equality to hold simultaneously for all t and ω from a ϑ invariant set of full measure. To achieve this we need a "perfection argument" (in the nontrivial case $\mathbb{T} = \mathbb{R}$).

Let μ resp. μ^+ be a probability measure on $\mathbb{R}$ resp. $\mathbb{R}^+$ which is equivalent to Lebesgue measure. Define

$$M := \{(t,\omega) \in \mathbb{R}^+ \times \Omega : \varphi(t,\omega)A(\omega) = A(\vartheta_t\omega)\}.$$

We have

$$\begin{aligned} 1 = (\mu^+ \times \mathbb{P})(M) &= \int_\Omega \int_{\mathbb{R}^+} 1_M(t,\omega)d\mu^+(t)d\mathbb{P}(\omega) \\ &= \int_{\mathbb{R}^+} \int_\Omega 1_M(t,\vartheta_{-t}\omega)d\mathbb{P}(\omega)d\mu^+(t), \end{aligned}$$

which implies $\mathbb{P}(\Omega_0) = 1$ with

$$\Omega_0 := \{\omega : (t,\vartheta_{-t}\omega) \in M \text{ for } \mu^+\text{–a.e. } t\}.$$

Since

$$(\mu \times \mathbb{P})\{(t,\omega) : \vartheta_t\omega \in \Omega_0\} = 1$$

the ϑ invariant set

$$\Omega_1 := \{\omega : \vartheta_t\omega \in \Omega_0 \text{ for } \mu\text{–a.e. } t\}$$

has also full measure.

Now assume $\omega \in \Omega_1$, $t \geq 0$ and $\vartheta_{-t}\omega \in \Omega_0$. For Lebesgue a.e. $s \geq t$ we have $(s-t, \vartheta_{-s}\omega) \in M$ and hence

$$\varphi(s, \vartheta_{-s}\omega)A(\vartheta_{-s}\omega) = \varphi(t, \vartheta_{-t}\omega)A(\vartheta_{-t}\omega) =: A^*(\omega).$$

The set $A^*(\omega)$ is uniquely defined by being equal to $\varphi(t,\omega)A(\vartheta_{-t}\omega)$ for Lebesgue almost all $t > 0$. In particular we have $A^*(\omega) = \varphi(t, \vartheta_{-t}\omega)A(\vartheta_{-t}\omega)$ for every $t \geq 0$ with $\vartheta_{-t}\omega \in \Omega_0 \cap \Omega_1$. The random compact set A^* is defined for $\omega \in \Omega_1$ and coincides with A on $\Omega_0 \cap \Omega_1$.

For $\omega \in \Omega_0 \cap \Omega_1$ we have already seen that $\varphi(t,\omega)A^*(\omega) = A^*(\vartheta_t\omega)$ for $t \geq 0$. For $\omega \in \Omega_1 \setminus \Omega_0$ choose $s > 0$ with $\vartheta_{-s}\omega \in \Omega_0$. For $t > 0$ we have then

$$\varphi(t,\omega)A^*(\omega) = \varphi(t+s, \vartheta_{-s}\omega)A(\vartheta_{-s}\omega) = A^*(\vartheta_t\omega),$$

i.e. A^* is invariant.

(2) Finally we show that A^* is an attractor for φ. Without loss of generality we assume $\Omega = \Omega_1$. Since $\mathbb{P}(A^*(\omega) = A(\omega)) = 1$ and A^* is invariant, A^* is an attractor for ψ.

Let $\varepsilon > 0$ and a random compact set $C \subset B$ be given. The continuity of $(t,x) \mapsto \varphi(t,\omega)x$ and $t \mapsto A^*(\vartheta_t\omega)$ (which follows from the invariance of A^*) implies that for every ω there exists $\delta(\omega) > 0$ with

$$\operatorname{dist}(x, A^*(\omega)) < \delta(\omega) \Rightarrow \operatorname{dist}(\varphi(t,\omega)x, A(\vartheta_t\omega)) < \varepsilon$$

for $0 \leq t < T$.

Choose $\delta_0 > 0$ with $\mathbb{P}(\delta(\omega) < \delta_0) < \frac{\varepsilon}{2}$ and $N \in \mathbb{N}$ with $\mathbb{P}(\operatorname{dist}(\Theta_\psi(n)C, A^*) \geq \delta_0) < \frac{\varepsilon}{2}$ for $n \geq N$. For $t = nT + s \geq NT$ with $n \in \mathbb{N}$ and $0 \leq s < T$ we have

$$\begin{aligned}\mathbb{P}(\operatorname{dist}(\Theta_\varphi(t)C, A^*) \geq \varepsilon) &\leq \mathbb{P}(\operatorname{dist}(\Theta_\psi(n)C, A^*) \geq \delta(\omega)) \\ &\leq \mathbb{P}(\operatorname{dist}(\Theta_\psi(n)C, A^*) \geq \delta_0) + \mathbb{P}(\delta(\omega) < \delta_0) < \varepsilon,\end{aligned}$$

which shows that A^* is a B attractor for φ. □

Remark 10. In the definition of a random attractor $\varphi(t,\omega)$ appears only for $t \geq 0$. Hence an attractor does not change if we restrict φ to $\mathbb{Z}^+ \times \Omega \times X$. However, invertibility of ϑ is needed.

3 Set Valued Random Dynamical Systems

In this section we deal (justified by Theorem 9) exclusively with discrete time $\mathbb{T} = \mathbb{Z}$. We will frequently omit the time argument in the time–one mappings, i.e. we write $\vartheta := \vartheta_1$, $\varphi(\omega) := \varphi(1, \omega)$ and $\Theta := \Theta(1)$.

The goal of this section is to investigate the robustness of random attractors under perturbations of the system. For this purpose we introduce the notion of a set valued random dynamical system and look for attractors for this type of systems. Our results can be applied to random dynamical systems defined by numerical schemes.

Definition 11. A (compact) *set valued random dynamical system* on a separable metric space X over (Ω, ϑ) is a measurable mapping

$$\varphi : \mathbb{N} \times \Omega \times X \to \mathcal{K}(X)^* := \mathcal{K}(X) \setminus \{\emptyset\}$$

with the following properties:

– φ is a cocyle, i.e.

$$\varphi(n+m, \omega)x = \varphi(m, \vartheta_n \omega)\varphi(n, \omega)x := \bigcup_{y \in \varphi(n,\omega)x} \varphi(m, \vartheta_n \omega)y$$

for $x \in X$, $n, m \in \mathbb{N}$ and $\omega \in \Omega$,

– $\varphi(n, \omega)$ is upper semi–continuous, i.e. $\lim_{i\to\infty} x_i = x$ implies $\lim_{i\to\infty} \operatorname{dist}(\varphi(n, \omega)x_i, \varphi(n, \omega)x) = 0$.

First we derive some basic properties of set valued mappings.

Lemma 12. *Let $K_n, L \subset X$ be compact sets with $\lim_{n\to\infty} \operatorname{dist}(K_n, L) = 0$. Then the sets $K := \overline{\bigcup_{n=1}^{\infty} K_n}$ and $K' := L \cup (\bigcup_{n=1}^{\infty} K_n)$ are compact.*

Proof. Let $(x_i)_{i\in\mathbb{N}} \subset K$ be any sequence. Then there exist sequences $(n(i)) \subset \mathbb{N}$ and $(y_i) \subset X$ with $y_i \in K_{n(i)}$ and $d(x_i, y_i) < \frac{1}{i}$.

First assume that there exists an n with $n(i) = n$ for infinitely many i. Then (y_i) has an accumulation point in $K_n \subset K$, which is also an accumulation point of (x_i).

Otherwise we have $\lim_{i\to\infty} n(i) = \infty$. In this case for every i there exists $z_i \in L$ with $\lim_{i\to\infty} d(y_i, z_i) = \lim_{i\to\infty} d(x_i, z_i) = 0$. Since L is compact, there exists an accumulation point z^* of (z_i), which is also an accumulation point of (x_i). The closedness of K implies $z^* \in K$. This shows that K is compact.

The compactness of K' follows because in both cases the accumulation point lies in K'. $\square$

Lemma 13. *Every upper semi–continuous map $\varphi : X \to \mathcal{K}(X)^*$ defines a mapping $\mathcal{K}(X)^* \to \mathcal{K}(X)^*$ also denoted by φ via $\varphi(K) = \bigcup_{x\in K} \varphi(x)$, which is also upper semi–continuous, i.e. $\lim_{i\to\infty} \operatorname{dist}(K_i, K) = 0$ implies $\lim_{i\to\infty} \operatorname{dist}(\varphi K_i, \varphi K) = 0$.*

Proof. In order to show that $\varphi(K)$ is compact let $(y_i) \subset \varphi(K)$ be a sequence. For every i there exists $x_i \in K$ with $y_i \in \varphi(x_i)$. Let x be an accumulation point of (x_i), w.l.o.g. $x = \lim x_i$. By Lemma 12 $\varphi(x) \cup (\bigcup_i \varphi(x_i)) \subset \varphi(K)$ is compact, which implies that (y_i) has an accumulation point in $\varphi(K)$.

The upper semi–continuity of φ on $\mathcal{K}(X)^*$ is proved by contradiction. Assume that there exist $\varepsilon > 0$ and sequences $(i(j))_j \subset \mathbb{N}$ and $(x_j) \subset X$ with $i(j) \uparrow \infty$, $x_j \in K_{i(j)}$ and $\operatorname{dist}(\varphi(x_{i(j)}, \varphi(K)) \geq \varepsilon$ for every j.

Since $\lim_{j\to\infty} \operatorname{dist}(K_{i(j)}, K) = 0$ the sequence (x_j) has an accumulation point $x \in K$, w.l.o.g. $x = \lim x_j$. We have

$$\operatorname{dist}(\varphi(x_{i(j)}), \varphi(K)) \leq \operatorname{dist}(\varphi(x_{i(j)}), \varphi(x)) \to 0,$$

which is the desired contradiction. □

Remark 14. (i) Lemma 13 implies that the composition of upper semi–continuous maps $X \to \mathcal{K}(X)^*$ defines again an upper semi–continuous map $X \to \mathcal{K}(X)^*$.

(ii) The upper semi–continuity of φ is equivalent to $\varphi\left(\bigcap_{i\in\mathbb{N}} K_i\right) = \bigcap_{i\in\mathbb{N}} \varphi(K_i)$ for every decreasing sequence $(K_i) \subset \mathcal{K}(X)^*$ (for the "$\Leftarrow$" direction consider $K_i := \{x\} \cup \{x_j : j \geq i\}$ for a sequence (x_j) with $x = \lim x_j$).

Corollary 15. *If (Ω, ϑ) is a metric dynamical system, X a separable metric space, and $\varphi : \Omega \times X \to \mathcal{K}(X)^*$ a measurable mapping which is upper semi–continuous in $x \in X$, then*

$$\varphi(n, \omega)x = \varphi(\vartheta_{n-1}\omega)...\varphi(\omega)x$$

defines a set valued random dynamical system on X over (Ω, ϑ).

Every set valued random dynamical system is generated in this way by its time–one mapping.

To proceed further we need some notations.

Definition 16. Let φ be a set valued random dynamical system on X over (Ω, ϑ).

(i) A random set $B \subset \Omega \times X$ is called *forward invariant* under φ, if $\varphi(\omega)x \subset B(\vartheta\omega)$ for every $x \in B(\omega)$.

(ii) A random compact set A is called *invariant* under φ, if $\varphi(\omega)A(\omega) = A(\vartheta\omega)$ $\mathbb{P}$–a.s.

(iii) Let $\mathcal{D}$ be a collection of random compact sets. A random compact set C is $\mathcal{D}$ *attracting*, if

$$\lim_{n\to\infty} \operatorname{dist}(\varphi(n, \omega)D(\omega), C(\vartheta_n\omega)) = 0$$

in probability for every $D \in \mathcal{D}$.

(iv) An invariant random compact set $A \subset B$ is a B *attractor*, if

$$\lim_{n\to\infty} \operatorname{dist}(\varphi(n, \omega)C(\omega), A(\vartheta_n\omega)) = 0$$

in probability for every random compact set $C \subset B$.

(v) A family $\mathcal{D}$ of random compact set *exhausts* a random set B, if for every random compact set $C \subset B$ and every $\varepsilon > 0$ there exists $D \in \mathcal{D}$ with $\mathbb{P}(C(\omega) \subset D(\omega)) > 1 - \varepsilon$.

Now we are prepared to prove a general criterion for the existence of an attractor for a set valued random dynamical system.

Theorem 17. *Let φ be a set valued random dynamical system, B a forward invariant random set and $\mathcal{D}$ a collection of random compact sets which exhausts B. Assume that there exists a random compact set $C \subset B$ which is $\mathcal{D}$ attracting with*

$$\lim_{n\to\infty} \operatorname{dist}(\Theta(n)C, C) = 0 \quad \mathbb{P}\text{-}a.s.$$

Then A defined by

$$A(\omega) := \bigcap_{n=1}^{\infty} \overline{\bigcup_{m\geq n} \varphi(m, \vartheta_{-m}\omega)C(\vartheta_{-m}\omega)}$$

is a B attractor.

Proof. (1) By Lemma 12 $A(\omega)$ is the intersection of a decreasing sequence of non–empty compact sets. Hence A is a non–empty random compact set. By assumption

$$\lim_{n\to\infty} \operatorname{dist}\left(\overline{\bigcup_{m\geq n} \varphi(m, \vartheta_{-m}\omega)C(\vartheta_{-m}\omega)},\, C(\omega)\right) = 0 \quad \mathbb{P}\text{–a.s.},$$

which implies $A \subset C \subset B$.

(2) Let $D \in \mathcal{D}$ and $\varepsilon > 0$ be given. The definition of A implies $\lim_{n\to\infty} \operatorname{dist}(\Theta(n)C, A) = 0$ $\mathbb{P}$–a.s., i.e. there exists $n_1 \in \mathbb{N}$ with $\mathbb{P}\left((\operatorname{dist}(\Theta(n_1)C, A) \geq \frac{\varepsilon}{2}\right) < \frac{\varepsilon}{3}$.

By the upper semi–continuity of $\varphi(n_1, \omega)$ and the compactness of $C(\omega)$ there exists $\delta(\omega) > 0$ with $\operatorname{dist}(\varphi(n_1,\omega)x, \varphi(n_1,\omega)C(\omega)) < \frac{\varepsilon}{2}$ if $\operatorname{dist}(x, C(\omega)) < \delta(\omega)$.

Choose $\delta_0 > 0$ with $\mathbb{P}(\delta(\omega) < \delta_0) < \frac{\varepsilon}{3}$. By assumption there exists $n_2 \in \mathbb{N}$ with $\mathbb{P}\left(\operatorname{dist}(\Theta(n)D, C) \geq \delta_0\right) < \frac{\varepsilon}{3}$ for $n \geq n_2$.

If $n \geq n_1 + n_2$, $\operatorname{dist}(\varphi(n - n_1, \omega)D(\omega), C(\vartheta_{n-n_1}\omega)) < \delta_0$, $\delta(\vartheta_{n-n_1}\omega) \geq \delta_0$ and $\operatorname{dist}(\varphi(n_1,\omega)C(\vartheta_{n-n_1}\omega), A(\vartheta_n\omega)) < \frac{\varepsilon}{2}$, then $\operatorname{dist}(\varphi(n,\omega)D(\omega), A(\vartheta_n\omega)) < \varepsilon$. Since this happens (for every $n \geq n_1 + n_2$) with probability $> 1 - \varepsilon$, we see that A is $\mathcal{D}$ attracting.

(3) Given $\varepsilon > 0$ and an arbitrary random compact set $E \subset B$ choose $D \in \mathcal{D}$ with $\mathbb{P}(E(\omega) \subset D(\omega)) > 1 - \frac{\varepsilon}{2}$ and $N \in \mathbb{N}$ with $\mathbb{P}(\operatorname{dist}(\Theta(n)D, A) > \varepsilon) < \frac{\varepsilon}{2}$ for $n \geq N$. Then for $n \geq N$

$$\mathbb{P}(\operatorname{dist}(\Theta(n)E, A) > \varepsilon) \leq \mathbb{P}(\operatorname{dist}(\Theta(n)D, A) > \varepsilon) + \mathbb{P}(E(\omega) \not\subset D(\omega)) < \varepsilon.$$

(4) It remains to show that A is invariant. The inclusion $A \subset \Theta A$ follows (using Lemma 12 and Remark 14.(ii)) from

$$\begin{aligned}\varphi(\omega)A(\omega) = \overline{\varphi(\omega)A(\omega)} &= \overline{\bigcap_{n=1}^{\infty} \varphi(\omega)\left(\overline{\bigcup_{m\geq n} \varphi(m,\vartheta_{-m}\omega)C(\vartheta_{-m}\omega)}\right)} \\ &\supset \overline{\bigcap_{n=1}^{\infty} \varphi(\omega)\left(\bigcup_{m\geq n} \varphi(m,\vartheta_{-m}\omega)C(\vartheta_{-m}\omega)\right)} \\ &= \bigcap_{n=1}^{\infty} \overline{\bigcup_{m\geq n} \varphi(\omega)\varphi(m,\vartheta_{-m}\omega)C(\vartheta_{-m}\omega)} = A(\vartheta\omega).\end{aligned}$$

Since $A \subset B$ and $\Theta(n)A \supset A$ for $n \geq 0$ we have for every $\varepsilon > 0$

$$\mathbb{P}(\operatorname{dist}(\Theta A, A) > \varepsilon) \leq \mathbb{P}(\operatorname{dist}(\Theta(n)A, A) > \varepsilon) \to 0 \text{ as } n \to \infty.$$

This implies $\varphi(\omega)A(\omega) \subset A(\vartheta\omega)$ with probability 1. □

Remark 18. (i) There are several versions of this theorem in the context of (standard) point valued random dynamical systems (e.g. Crauel, Debussche and Flandoli [11]). The most general one is due to Keller and Schmalfuß [27, Theorem 2.2].

There is also an extension to set valued non–autonomous systems (where the attractor is defined via "pullback convergence") by H. Keller (unpublished). The case where C is an absorbing set (also in the framework of set valued non–autonomous systems) was treated by Kloeden and Schmalfuß [30].

(ii) If there exists a forward invariant random compact set C, then $A = \bigcap_{n\geq 0} \Theta(n)C$ is a C attractor. This particular case of Theorem 17 will be important for our applications (see Theorem 19 and Corollary 23).

The following result applies to robustness of attractors under small perturbations.

Theorem 19. *Let φ_k, φ ($k \in \mathbb{N}$) be set valued random dynamical systems on X over (Ω, ϑ) with $\varphi(\omega)x \subset \varphi_{k+1}(\omega)x \subset \varphi_k(\omega)x$ for every $k \in \mathbb{N}$, $\omega \in \Omega$ and $x \in X$. Assume that there exists a random compact set B, which is forward invariant under φ_1 (and hence under all φ_k). With $\operatorname{dist}_B(\varphi_k, \varphi)(\omega) := \sup_{x\in B(\omega)} \operatorname{dist}(\varphi_k(\omega)x, \varphi(\omega)x)$ suppose $\lim_{k\to\infty} \operatorname{dist}_B(\varphi_k, \varphi) = 0$ $\mathbb{P}$–a.s.*

Then every φ_k has a B attractor A_k with $A_{k+1} \subset A_k$, and $A := \bigcap_{k\in\mathbb{N}} A_k$ is a B attractor for φ.

Proof. The existence of B attractors A_k and A' for φ_k resp. φ is clear from Theorem 17. Obviously we have $A' \subset A_{k+1} \subset A_k$ for every k, which implies $A' \subset A$. By Remark 14.(ii)

$$\varphi(\omega)A(\omega) = \bigcap_{k\in\mathbb{N}} \varphi(\omega)A_k(\omega) \subset \bigcap_{k\in\mathbb{N}} \varphi_k(\omega)A_k(\omega) = A(\vartheta\omega).$$

Since

$$\operatorname{dist}(A_k(\vartheta\omega), \varphi(\omega)A_k(\omega)) \leq \operatorname{dist}(\varphi_k, \varphi) \to 0$$

it follows $\bigcap_{k\in\mathbb{N}} \varphi(\omega)A_k(\omega) = A(\vartheta\omega)$, i.e. A is invariant under φ.

Since A' is an attractor we have

$$\mathbb{P}(\operatorname{dist}(A, A') > \varepsilon) = \mathbb{P}(\operatorname{dist}(\Theta_\varphi(n)A, A') > \varepsilon) \to 0$$

as $n \to \infty$ for every $\varepsilon > 0$, which implies $A \subset A'$ $\mathbb{P}$–a.s. □

Corollary 20. *Let φ, φ_k ($k \in \mathbb{N}$) be set valued random dynamical systems. Assume that there exists a random compact set B which is forward invariant under all φ_k.*

If $\lim_{k\to\infty} \operatorname{dist}_B(\varphi_k, \varphi) = 0$, then there exist B attractors A and A_k for φ resp. φ_k with $\lim_{k\to\infty} \operatorname{dist}(A_k, A) = 0$ $\mathbb{P}$–a.s.

Proof. First observe that the compactness of $B(\omega)$ and the forward invariance of B under all φ_k imply, that B is forward invariant under φ.

Define

$$\psi_k(\omega)x := \varphi(\omega)x \cup \left(\bigcup_{l\geq k} \varphi_l(\omega)x\right).$$

We have to show that ψ_k, φ satisfy the assumptions of Theorem 19. The inclusions $\varphi(\omega)x \subset \psi_{k+1}(\omega)x \subset \psi_k(\omega)x$ and $\psi_1(\omega)B(\omega) \subset B(\vartheta\omega)$ are clear. Furthermore, $\operatorname{dist}_B(\psi_k, \varphi) \to 0$.

By Lemma 12, $\psi_k(\omega)x$ is compact for every k, ω, x. To show that $\psi_k(\omega)$ is upper semi–continuous consider sequences $(x_n) \subset B(\omega)$ with $x = \lim x_n$ and (y_n) with $y_n \in \psi_k(\omega)x_n$ and $y = \lim y_n$. We have to show $y \in \psi_k(\omega)x$.

This follows from the upper semi–continuity of $\varphi(\omega)$ in the case that $y_n \in \varphi(\omega)x_n$ for infinitely many n. Otherwise there exist $l(n) \geq k$ (for all but finitely many n) with $y_n \in \varphi_{l(n)}(\omega)x_n$. If $\lim_{n\to\infty} l(n) = \infty$, then

$$\begin{aligned}
&\operatorname{dist}(\varphi_{l(n)}(\omega)x_n, \varphi(\omega)x) \\
&\leq \operatorname{dist}(\varphi_{l(n)}(\omega)x_n, \varphi(\omega)x_n) + \operatorname{dist}(\varphi(\omega)x_n, \varphi(\omega)x) \to 0
\end{aligned}$$

as $n \to \infty$, which implies $y \in \varphi(\omega)x \subset \psi_k(\omega)x$.

If $l(n) \not\to \infty$, then there exists $l \geq k$ with $l(n) = l$ for infinitely many n, which implies $y \in \varphi_l(\omega)x \subset \psi_k(\omega)x$. □

Remark 21. (i) There is some related work in the context of non–autonomous systems by Kloeden and Kozyakin [28,29].

(ii) Corollary 20 implies in particular to point valued random dynamical systems φ_k, φ with $\lim_{k\to\infty} \sup_{x\in B(\omega)} d(\varphi_k(\omega)x, \varphi(\omega)x) = 0$ $\mathbb{P}$–a.s., if there exists a random compact set B which is invariant under all φ_k.

The previous results only ensure the persistence of local ore relative attractors. However, the next result shows that the A_k are global attractors if B is an absorbing set for φ_k.

Proposition 22. *Let φ be a set valued random dynamical system, B a forward invariant random set with $B(\omega)$ open and A a B attractor.*

Then A is a $\tilde{B}$ set attractor for φ with

$$\tilde{B} := \{(\omega, x) : \varphi(n, \omega)x \subset B(\vartheta_n \omega) \text{ for some } n \geq 0\}.$$

Proof. Let $\mathcal{D}$ be the family of all random compact sets D for which $\Theta(n)D \subset B$ for some $n \geq 0$. Then A is $\mathcal{D}$ attracting.

By Theorem 17 it remains to show that $\mathcal{D}$ exhausts $\tilde{B}$. Let $C \subset \tilde{B}$ be a random compact set. For $x \in C(\omega)$ there exists a minimal $n = n(\omega, x) \geq 0$ with $\varphi(n, \omega)x \subset B(\vartheta_n \omega)$. The upper semi–continuity of $\varphi(n, \omega)$ implies $n(\omega, y) \leq n(\omega, x)$ for y from an open neighborhood of x. By compactness of $C(\omega)$ it follows $n(\omega) := \sup_{x \in C(\omega)} n(\omega, x) < \infty$.

Given $\varepsilon > 0$ there exists $n \geq 0$ with $\mathbb{P}(n(\omega) > n) < \varepsilon$. Define $D(\omega) := C(\omega)$ if $n(\omega) \leq n$ and $D(\omega) := A(\omega)$ if $n(\omega) > n$. Then $D \in \mathcal{D}$ and $\mathbb{P}(C(\omega) \subset D(\omega)) > 1 - \varepsilon$. □

Corollary 23. *Let φ be a set valued random dynamical system on X and C a random compact set with $\varphi(\omega)C(\omega) \subset \mathrm{int}C(\vartheta\omega)$. Assume C is absorbing, i.e. for every $x \in X$ and (almost) every $\omega \in \Omega$ there exists $n \in \mathbb{N}$ with $\varphi(n, \omega)x \in C(\vartheta_n \omega)$.*

Then $A := \bigcap_{n=1}^{\infty} \Theta(n)C$ is the global attractor for φ.

Proof. Apply Remark 18.(ii) and Proposition 22 with $B(\omega) = \mathrm{int}C(\omega)$. □

We now give a criterion for the robustness of a global attractor for a random dynamical system φ on $\mathbb{R}^d$, which imposes conditions only on φ and works for a general type of perturbations. It is based on the existence of a "Lyapunov type" function, which can be verified in many concrete examples.

Theorem 24. *Let φ be a discrete time (point valued) random dynamical system on $\mathbb{R}^d$ over (Ω, ϑ) and $V : \mathbb{R}^d \to \mathbb{R}^+$ a continuous "Lyapunov function" with $\lim_{|x| \to \infty} V(x) = \infty$. With*

$$\eta_c(x) := \sup_{|y-x| \leq c} \frac{|V(y) - V(x)|}{V(x)}$$

assume $\lim_{|x| \to \infty} \eta_c(x) = 0$ for some $c > 0$. Furthermore suppose that there exist measurable functions $a, b : \Omega \to \mathbb{R}^+$ with $\int \log^+ a \, d\mathbb{P}, \int \log^+ b \, d\mathbb{P} < \infty$ and $\int \log a \, d\mathbb{P} < 0$, such that

$$V(\varphi(\omega)x) \leq a(\omega)V(x) + b(\omega)$$

for $\omega \in \Omega$ and $x \in \mathbb{R}^d$ ($\log^+ \xi := \max\{0, \log \xi\}$).

Let $(\varphi_k)_{k\in\mathbb{N}}$ be a sequence of set valued random dynamical systems on $\mathbb{R}^d$ over (Ω,ϑ) with $\operatorname{dist}(\varphi_k(\omega)x,\varphi(\omega,x)) \leq c$ *for* $x \in \mathbb{R}^d$ *and* $\lim_{k\to\infty}\operatorname{dist}(\varphi_k(\omega)x,\varphi(\omega,x)) = 0$ $\mathbb{P}$*–a.s. for every* $x\in\mathbb{R}^d$.

Then φ has a global attractor A and every φ_k has a global attractor A_k with $\lim_{k\to\infty}\operatorname{dist}(A_k,A) = 0$ $\mathbb{P}$*–a.s.*

If in addition $\varphi(\omega)x \in \varphi_k(\omega)x$ for $x\in\mathbb{R}^d$ and $k\in\mathbb{N}$, then $A\subset A_k$ for every k and $\lim_{k\to\infty} A_k = A$ $\mathbb{P}$–a.s. in the Hausdorff metric.

Proof. Define a set valued random dynamical system ψ by

$$\psi(\omega)x := \{y : |y-\varphi(\omega)x| \leq c\}.$$

By Corollary 20 and Corollary 23 it suffices to construct an absorbing random compact set C for ψ with $\psi(\omega)C(\omega) \subset \operatorname{int}C(\vartheta\omega)$.

Choose $\varepsilon > 0$ with $\int \log\left(\frac{a}{1-\varepsilon}+\varepsilon\right) d\mathbb{P} < 0$ and b_0 with $\eta_c(x) \leq \varepsilon$ for $|x| \geq b_0$. Set

$$\tilde b(\omega) := \max\left\{\frac{b(\omega)}{1-\varepsilon}, b_0\right\},$$

$a_1(\omega) := \frac{a(\omega)}{1-\varepsilon}$ and $a_2(\omega) := a_1(\omega)+\varepsilon$. Then $\int\log^+ a_{1,2}\,d\mathbb{P}, \int\log^+\tilde b\,d\mathbb{P} < \infty$ and $\int\log a_{1,2}\,d\mathbb{P} < 0$.

By $\chi_{1,2}(\omega)v := a_{1,2}(\omega)v + \tilde b(\omega)$ affine random dynamical systems $\chi_{1,2}$ on $\mathbb{R}$ over (Ω,ϑ) are defined. By [1, Theorem 5.6.5, Corollary 5.6.6] there exist random variables $v_1, v_2 : \Omega\to\mathbb{R}$ with $\chi_i(\omega)v_i(\omega) = v_i(\vartheta\omega)$, such that $(\{v_i(\omega)\})_{\omega\in\Omega}$ is the global attractor for χ_i. This implies with $a_i(\omega)\geq 0$ and $\tilde b(\omega) \geq b_0$ that $b_0 \leq v_1(\omega) \leq v_2(\omega) - \varepsilon b_0$

Assume $V(x) \leq v_2(\omega)$, $y\in\psi(\omega)x$ and $V(y) \geq v_2(\vartheta\omega)$. Then by assumption

$$\begin{aligned} &V(y) \leq V(\varphi(\omega)x) + \eta_c(y)V(y) \leq a(\omega)V(x) + b(\omega) + \varepsilon V(y)\\ \Rightarrow\ &V(y) \leq a_1(\omega)v_2(\omega) + \tilde b(\omega) < v_2(\vartheta\omega), \end{aligned}$$

which is a contradiction, i.e. with $C(\omega) := \{x : V(x) \leq v_2(\omega)\}$ we have $\psi(\omega)C(\omega) \subset \operatorname{int}C(\vartheta\omega)$.

In order to show that C is absorbing assume $v := V(x) \geq v_2(\omega)$. It follows from [1, Corollary 5.6.6.(ii)] that there exists $n\geq 0$ with $|\chi_1(n,\omega)v - v_1(\vartheta_n\omega)| < \varepsilon b_0$. Set $x_0 = 0$ and for $k = 1,...,n$ let $x_k \in \varphi(\vartheta_{k-1}\omega)x_{k-1}$. If $V(x_k) < v_2(\vartheta_k\omega)$ for some $k\leq n$ then $x_n \in \operatorname{int}C(\vartheta_n\omega)$.

Now suppose $V(x_k) \geq v_1(\vartheta_k\omega) \geq b_0$ for $0\leq k\leq n$. Then

$$V(x_k) \leq a_1(\omega)V(x_{k-1}) + \tilde b(\omega).$$

Inductively it follows $V(x_n) \leq \chi_1(n,\omega)v < v_2(\vartheta_n\omega)$, which implies $\psi(n,\omega)x \subset C(\vartheta_n\omega)$. □

Remark 25. If φ is a continuous time random dynamical system with $t \mapsto \varphi(t,\omega)x$ differentiable and V differentiable with

$$\frac{\partial}{\partial t} V(\varphi(t,\omega)x) \leq \alpha(\vartheta_t\omega) V(\varphi(t,\omega)x) + \beta(\vartheta_t\omega)$$

with $\alpha, \beta \in L^1(\mathbb{P})$, $t \mapsto \alpha(\vartheta_t\omega), \beta(\vartheta_t\omega)$ continuous and $\int \alpha \, d\mathbb{P} < 0$, then the time–one mapping of φ meets the assumptions of Theorem 24. This follows by considering the continuous time affine random dynamical system χ generated by the random differential equation

$$\dot{v}(t) = \alpha(\vartheta_t\omega)v(t) + \beta(\vartheta_t\omega)$$

(cf. [1, Theorem 5.6.5, Corollary 5.6.6]). Then $V(\varphi(1,\omega)x) \leq \chi(1,\omega)V(x)$.

4 Numerics of Random Attractors and Invariant Measures

4.1 Attractors

The *subdivision algorithm* of Dellnitz and Hohmann [14,15] was adapted to the case of random dynamical systems by Keller and Ochs [25]. Here we give only a brief description of the algorithm. For details of the implementation consult [14,15,25].

Given a discrete time random dynamical system φ on $\mathbb{R}^d$ over (Ω, ϑ) the algorithm defines a set valued random dynamical system $\hat{\varphi}$ in the following way. The state space $\mathbb{R}^d$ is divided into small (closed) boxes, which intersect only at their boundaries. The image of a point x under $\hat{\varphi}(\omega)$ consists of all those boxes, which contain $\varphi(\omega)x$. Then $\varphi(\omega)x \in \hat{\varphi}(\omega)x$ and $\mathrm{dist}(\varphi(\omega)x, \hat{\varphi}(\omega)x)$ is at most the diameter of the small boxes.

The boxes can be subdivided by bisection with respect to one coordinate (which changes cyclically between successive subdivisions). This gives a sequence $(\hat{\varphi}_k)_{k\geq 0}$ of "numerical" mappings each corresponding to a certain box size, i.e. $\hat{\varphi}_k$ describes the numerical mapping after k subdivisions. By construction

$$\lim_{k\to\infty} \sup_{\omega\in\Omega} \sup_{x\in\mathbb{R}^d} \mathrm{dist}(\varphi(\omega)x, \hat{\varphi}_k(\omega)x)) = 0.$$

With Theorem 24 this gives

Corollary 26. *Let the random dynamical system φ satisfy the assumptions of Theorem 24 (which implies the existence of a global attractor A).*

Then the set valued random dynamical systems $\hat{\varphi}_k$ defined by the numerical discretization have global attractors A_k with $A \subset A_k$ and $\lim_{k\to\infty} A_k = A$ $\mathbb{P}$–a.s. in the Hausdorff metric.

In practise it is of course not possible to treat the whole $\mathbb{R}^d$ (which consists of infinitely many boxes). We start with one "sufficiently large" box Q, which is then successively subdivided into smaller boxes. The choice of Q should ensure $A(\omega) \subset Q$ with "high" probability. We set $Q_0 = Q$ and for $n \geq 0$ we define Q_{n+1} to be the part of Q which belongs to the numerical image of Q_n under the map $\hat{\varphi}_k(\vartheta_n\omega)$ for some $k(n)$. Typically we start with $k(0)$ small (often $= 0$) and increase $k(n)$ by subdivisions in the course of the simulation. After a certain number of subdivisions we stay at a constant level, i.e. $k(n) = k$ for all $n \geq n_0$ with some $k, n_0 \in \mathbb{N}$.

If $A(\omega) \subset Q$ and

$$\hat{\varphi}_k(\vartheta_i\omega)Q_i \subset Q \qquad \text{for } i = 0, ..., n-1 \tag{$*$}$$

(note that $(*)$ is numerically checkable), then we know $A(\vartheta_n\omega) \subset Q_n$. As $n \to \infty$ our approximations Q_n should "approach" $A_k(\vartheta_n\omega)$, which is close to $A(\vartheta_n\omega)$ if k is sufficiently large.

If Q is sufficiently large, then the dynamics is expected to point inward, i.e. we have a good chance for $(*)$ to be fulfilled. In case $(*)$ is violated it is possible to change the size (and position in space) of Q adaptively, i.e. we still can ensure to have a covering of the attractor. However, we do not have general estimates on $\mathrm{dist}(Q_n, A(\vartheta_n\omega))$ (these would involve additional assumptions on the "speed of attraction" of trajectories of φ towards A).

4.2 Invariant Measures

Dellnitz and Junge [16] approximate "natural" invariant measures for deterministic dynamical systems by calculating eigenvectors of a discretized Perron–Frobenius operator, i.e. they just discretize the equation $f\mu = \mu$ where $f : \mathbb{R}^d \to \mathbb{R}^d$ is a diffeomorphism and μ a probability measure supported by the attractor of f. Their space discretization is based on the box covering of the attractor produced by the subdivision algorithm. The discretized measure assigns a mass to each box, i.e it corresponds to a vector in $\mathbb{R}^N$ with non–negative entries, where N is the total number of boxes. The action of f on probabilities is approximated by a linear mapping on $\mathbb{R}^N$ given by a matrix $M(f)$ with non–negative entries. The entry corresponding to a pair (B, C) of boxes is the relative Lebesgue measure of that part of B which is mapped into C. That is, the space discretization gives a positive linear mapping $M(f) : \mathbb{R}^N \to \mathbb{R}^N$, which has (under an irreducibility and aperiodicity condition) by the Perron–Frobenius theorem a unique non–negative normed eigenvector corresponding to the eigenvalue one. Dellnitz and Junge show, that if f has a hyperbolic attractor A, then this eigenvector is an approximation of the Sinai–Ruelle–Bowen (SRB) measure on A. This SRB measure is the natural invariant measure for f in the sense that it describes the statistical behavior of the trajectories for Lebesgue almost all initial values.

What happens if we look for a generalization to the random case? Assume we have a differentiable discrete time random dynamical system φ (which may be the time–T mapping of a stochastic or random differential equation) with a global random attractor A. The numerical discretization gives a box covering of $A(\omega)$ with a random number $N(\omega)$ of boxes. Positive linear mappings $M(\varphi(\omega)) : \mathbb{R}^{N(\omega)} \to \mathbb{R}^{N(\vartheta\omega)}$ can be defined in the same way as in the deterministic case. There is a "random" Perron–Frobenius theorem by Gundlach and Steinkamp [21], which covers this situation. That is, under an irreducibility and an aperiodicity condition there exists a unique random vector $a(\omega) \in \mathbb{R}^{N(\omega)}$ with non–negative entries which sum up to one, and $M(\varphi(\omega))a(\omega) = a(\vartheta\omega)$. If we consider the linear random dynamical system Φ generated by $M(\varphi(\cdot))$, then $\lim_{n\to\infty} |\Phi(n,\omega)v - a(\vartheta_n\omega)| = 0$ for every probability vector $v \in \mathbb{R}^{N(\omega)}$.

However, the approach of Dellnitz and Junge is not directly transferable to the random case. Our "random eigenvector equation" $\Phi(\omega)a(\omega) = a(\vartheta\omega)$ involves ω and is thus an infinite dimensional problem, for which there is no canonical discretization, since we typically have little regularity in ω.

For this reason our approach is more primitive than the one of Dellnitz and Junge. We assign a mass to each box of the covering (starting with a uniform distribution) and just iterate this mass forward, i.e. we apply the map $\Phi(\omega)$. We expect our approximation to "converge" to $a(\vartheta_n\omega)$ for n sufficiently large. If the irreducibility and/or the aperiodicity condition of the random Perron–Frobenius theorem is violated, then Φ has still a random attractor in the space of non–negative normed vectors, which should be approached by our iterations.

However, there are no rigorous convergence results available up to now. First we have to ask which object $a(\omega)$ should approximate. There is a theory of SRB measures for random dynamical systems [5,31], which are characterized by satisfying Pesin's entropy formula and by absolute continuity on unstable manifolds. If the Markov process generated by an SDE has an absolutely continuous invariant measure, then (under an integrability condition) the corresponding Markov measure for the random dynamical system is a random SRB measure [31, Chapter IV, Theorem 1.1].

There is a possibility to prove convergence of our numerical approximation to the Markov measure using Proposition 4. However, this would not give a really satisfactory result. Therefore we will only sketch the idea instead of giving a complete proof. If the measure ρ in Proposition 4 is absolutely continuous (which is typically the case), then we can replace it by another absolutely continuous measure, namely the uniform distribution on our initial box (if $\Phi(\cdot)$ is not aperiodic, we have to be a little bit more careful). Then we can argue in the same way as in the proof of [25, Theorem 3.1]. The result would be the following:

Given ω and $\varepsilon > 0$ there exists $\delta > 0$ and $n \in \mathbb{N}$ such that the Prohorov distance between the probability measure on $\mathbb{R}^d$ represented by $\Phi(\vartheta_{-n}\omega)v$ and μ_ω is smaller than ε provided the box diameter is smaller than δ. Here v

is the probability vector representing the uniform distribution on the initial box Q.

Note that v is not concentrated on the covering of $A(\vartheta_{-n}\omega)$ but on the whole set Q. This means that (if δ is small) v has a huge number of entries, which is in practise not possible to handle.

A more satisfactory approach would be to relate the attractor of Φ (which always exists, since Φ leaves the random compact set of probability vectors on $\mathbb{R}^{N(\omega)}$ invariant) with the Markov measure (which is an attractor on a subset of the space of all probability measures).

5 The Stochastically Forced Duffing Oscillator

5.1 Introductory Remarks

The second order ODE

$$\ddot{x} = f(x) - \gamma\dot{x}, \quad x \in \mathbb{R},$$

describes a damped oscillator with a single degree of freedom, where $f(x) = -U'(x)$ for some (differentiable) potential function $U : \mathbb{R} \to \mathbb{R}$. If there is some damping (i.e. γ is positive), the system looses energy (which is given by $E(x, \dot{x}) = U(x) + \frac{\dot{x}^2}{2}$).

This implies in the case $\lim_{x\to\pm\infty} U(x) = \infty$ and $\{x : f(x) = 0\}$ finite the existence of a global attractor for the corresponding dynamical system on $\mathbb{R}^2$. The attractor contains all equilibria which correspond to the zeros of f (and which are stable for local minima of U).

The *Duffing oscillator* has the potential $U(x) = \frac{\beta}{4}x^4 - \frac{\alpha}{2}x^2$ with $\alpha, \beta > 0$. Hence there exists a global attractor which contains 3 equilibria, two stable ones (with $x = \pm\sqrt{\frac{\alpha}{\beta}}$, $\dot{x} = 0$) and a saddle in the origin.

If a periodic forcing is introduced, the equation (written as a first order system with $y = \dot{x}$) reads

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \alpha x - \beta x^3 - \gamma y + \sigma\cos(\mu t) \end{aligned}$$

with additional parameters $\sigma, \mu \in \mathbb{R}$. This equation can be interpreted as a random differential equation on $\mathbb{R}^2$ with $\Omega = S^1 = \{z \in \mathbb{C} : |z| = 1\}$ and $\vartheta_t\omega = \exp(2\pi i\mu t)\omega$. Then the "noise term" takes the form $\cos(\mu t) = \Re(\vartheta_t\omega_0)$ with $\omega_0 = 1 \in \Omega$.

It can be shown that this system possesses a global "random" attractor, which looks for small σ qualitatively like the attractor of the unforced system. However, if σ increases, the dynamics becomes considerably more complicated (for a detailed description see Guckenheimer and Holmes [19, Chapter 2.2]). This is due to a transversal intersection of the stable and the unstable manifold of the "perturbed origin" (the hyperbolic fixed point persists if the perturbation does not exceed a certain size).

We go beyond this periodic forcing and consider a forcing term given by a stochastic differential dW, i.e. we have the equation

$$\begin{aligned} dx &= y\,dt \\ dy &= (f(x) - \gamma y)\,dt + c(x) \circ dW \end{aligned} \tag{$*$}$$

with $f = -U'$ and $c : \mathbb{R} \to \mathbb{R}$. In particular we will focus on the case $f(x) = \alpha x - \beta x^3$ and $c(x) \equiv \sigma$.

The corresponding Markov process has a unique invariant measure ρ with density

$$c \exp\left(\frac{-1}{\eta} E(x,y)\right) = c \exp\left(\frac{-1}{\eta}\left(U(x) + \frac{y^2}{2}\right)\right)$$

with respect to Lebesgue measure, where $c > 0$ is a normalization constant and $\eta = \frac{\sigma^2}{2\gamma}$.

Note that in particular this measure is independent of γ if we choose $\sigma = \sqrt{2\eta\gamma}$ with fixed $\eta > 0$. This situation in the Markov setup was considered by Schimansky–Geier and Herzel [35]. They observed numerically, that the top Lyapunov exponent changes its sign under variation of γ. This indicates a change of the qualitative dynamical behavior, which cannot be observed in terms of the invariant measure for the Markov process. For us this served as a motivation to investigate the stochastically forced Duffing oscillator in the framework of random dynamical systems.

5.2 Existence of a Global Random Attractor

It is quite hard to get estimates in order to show the existence of a random attractor directly from an SDE. A way out is to apply a random coordinate transformation which turns the SDE into a random differential equation. Such a transformation is possible for a large number (possibly for all) SDE's on $\mathbb{R}^d$ [22]. For a random differential equation it is often possible to obtain estimates of the type of Remark 25. If the transformed equation has an attractor, then it can be transformed back to an attractor for the original system.

This technique has been used by Keller and Schmalfuß [26] to verify the existence of a global attractor for the stochastically forced Duffing–van der Pol equation. It has been generalized by Imkeller and Schmalfuß [23]. There are also applications to stochastic partial differential equations [24,27].

In [23, Sect. 3.4] the existence of a global attractor for the Duffing equation with additive noise (i.e. $c(x) \equiv \sigma$) is proved. We will present a slight generalization of this result to the case if the potential U is an arbitrary polynomial with $\lim_{|x|\to\infty} U(x) = \infty$. However, our main purpose in this part of the paper is just to demonstrate the essential techniques which can be used to prove the existence of a global attractor for an SDE. We restrict our attention to a situation which is technically not so complicated rather than proving our result in most possible generality.

The following considerations (up to Theorem 27) are mainly based on ideas of H. Keller. They are applicable in more general situations (e.g. with non–constant $c(x)$, see [24]). However, then the calculations become considerably more elaborate.

In the first part (where we stay more general) we construct the coordinate transformation. Afterwards we present an estimate which ensures the existence of an attractor for the transformed equation.

Consider (as an auxiliary tool) the SDE $dz = -\mu z + \circ dW$ with parameter $\mu > 0$. This equation generates the so called *Ornstein–Uhlenbeck process.*

The generated random dynamical system χ has a unique stationary solution $z(\omega)$ (i.e. $\chi(t,\omega)z(\omega) = z(\vartheta_t\omega)$) [1, Theorem 5.6.5], which is the global attractor for χ [1, Corollary 5.6.6]. Note that χ is defined over the path space $\Omega = \{\omega \in C(\mathbb{R},\mathbb{R}) : \omega(0) = 0\}$ of the Wiener process. The random variable z has the explicit form $z(\omega) = -\mu \int_{-\infty}^{0} e^{\mu\tau}\omega(\tau)d\tau$ [18,26]. $t \mapsto z(\vartheta_t\omega)$ is continuous and $z(\omega)$ is normally distributed with mean 0 and variance $\frac{1}{2\mu}$.

Consider now Equation $(*)$ and set $v := y - c(x)z + \varepsilon x$ with $0 \le \varepsilon < \gamma$. Suppose $c \in C^2(\mathbb{R},\mathbb{R})$ with uniformly bounded derivative c'. Then

$$dx = y\,dt = (v + c(x)z - \varepsilon x)\,dt$$

and (using Îto's rule [18] and going a little bit sloppy over the technical details of the stochastic calculus)

$$\begin{aligned} dv &= dy - c(x)dz - zc'(x)\,dx - \varepsilon dx \\ &= (f(x) - \gamma(v + c(x)z - \varepsilon x))\,dt + c(x)\circ dW \\ &\quad -c(x)(-\mu z\,dt + \circ dW) - (c'(x)z + \varepsilon)(v - c(x)z - \varepsilon x)\,dt. \end{aligned}$$

In particular we observe that the "stochastic" terms $\pm c(x)\circ dW$ cancel. Hence our transformed equation is a random differential equation (with right hand side continuous in (t,x) and differentiable in x)

$$\begin{aligned} \dot{x} &= v - \varepsilon x + c(x)z \\ \dot{v} &= f(x) - (\gamma - \varepsilon)(v - \varepsilon x) \\ &\quad +(\mu + \varepsilon - \gamma)c(x)z - c'(x)z(v - \varepsilon x + c(x)z). \end{aligned} \tag{$**$}$$

We are free to vary the parameters ε and μ (the latter enters also into the distribution of the random variable z) in order to get some estimates.

For simplicity we assume from now on that $c(x)$ is equal to a constant $\sigma > 0$. Hence the $c'(x)$ term in $(**)$ vanishes. The equation takes the simplest possible form if we set $\varepsilon = 0$ and $\mu = \gamma$. Then the noise is "shifted" from "white noise" in the y coordinate to "real noise" in the x coordinate.

However, our purpose is to ensure that the random dynamical system φ generated by $(**)$ satisfies the assumptions of Theorem 24 (resp. Remark 25) with the the "energy" $E(x,v) = U(x) + \frac{v^2}{2}$ as a "Lyapunov function". This will only be possible if we choose ε with $0 < \varepsilon < \gamma$. For simplicity we set $\mu := \gamma - \varepsilon$. By eventual addition of a constant we can assume $E > 0$. We have

$$\frac{\partial}{\partial t} E(x(t), v(t)) = U'(x(t))\dot{x}(t) + v(t)\dot{v}(t), \text{ i.e.}$$

$$\begin{aligned}\dot{E} &= -f(x)(v - \varepsilon x + \sigma z) + v(f(x) - (\gamma - \varepsilon)(v - \varepsilon x) + \sigma(\mu + \varepsilon - \gamma)z) \\ &= -\varepsilon x U'(x) - (\gamma - \varepsilon)v^2 + \varepsilon(\gamma - \varepsilon)xv - \sigma z f(x) \\ &=: g(x,v,z).\end{aligned}$$

Now we make another restriction, namely that the potential U is a polynomial in x of even degree ≥ 2. Since $|ab| \leq \frac{1}{2}(a^2 + b^2)$ for real numbers a, b we have

$$\varepsilon(\gamma - \varepsilon)xv \leq \frac{1}{2}(\gamma - \varepsilon)v^2 + \varepsilon\frac{\varepsilon(\gamma - \varepsilon)}{2}x^2.$$

If we fix $\varepsilon > 0$ sufficiently small, this implies

$$\liminf_{|(x,v)| \to \infty} \frac{xU'(x) + v^2 - \varepsilon(\gamma - \varepsilon)xv}{E(x,v)} > 0.$$

Furthermore, by our assumption on U,

$$\lim_{|(x,v)| \to \infty} \frac{f(x)}{E(x,v)} = 0.$$

Hence we can find constants $R, \alpha_0, \delta > 0$ with

$$g(x,v,z) \leq (-\alpha_0 + \delta|z|)E(x,v)$$

if $|(x,v)| \geq R$. By increasing R we get δ arbitrarily small without changing α_0. Since z is normally distributed $|z|$ has finite expectation, i.e. we can choose R and δ with $\delta \int |z| \, d\mathbb{P} < \alpha_0$. Then

$$\begin{aligned}\alpha(\omega) &:= \delta|z(\omega)| - \alpha_0 \quad \text{and} \\ \beta(\omega) &:= \sup_{|(x,v)| \leq R} (\alpha_0 E(x,v) + g(x,v,z(\omega)))\end{aligned}$$

satisfy the conditions of Remark 25 after Theorem 24 with $V = E$. Hence we have

Theorem 27. *Let $U : \mathbb{R} \to \mathbb{R}^+$ be a polynomial of even degree ≥ 2 with $\lim_{x \to \infty} U(x) = \infty$ and $f(x) = -U'(x)$.*

Then the random dynamical system φ generated by the SDE (*) with $c(x) \equiv \sigma > 0$ (additive noise case) possesses a global random attractor.

Furthermore, φ is related to the random dynamical system ψ generated by (**) via a linear random coordinate transformation. The system ψ has for sufficiently small $\varepsilon > 0$ also a random attractor, which is robust under perturbations in the sense of Theorem 24.

Proof. It remains to show that the existence of an attractor for ψ implies the existence of an attractor for φ. This follows from [32, Theorem 4] (and is not very hard to be observed directly). □

Remark 28. The restriction on ε is due to our proof and can be removed with little technical effort.

5.3 Numerics and Discussion

Consider the random dynamical system φ generated by $(*)$ with $f(x) = \alpha x - \beta x^3$ and $c(x) \equiv \sigma$ with parameters $\alpha, \beta, \gamma, \sigma > 0$. We expect two regimes with different qualitative dynamics depending on the sign of the top Lyapunov exponent λ_1 of the Markov measure (the second exponent λ_2 as well as $\lambda_1 + \lambda_2$ are always negative since the system is volume contracting).

We have calculated λ_1 numerically (this is done by just calculating the expansion rate of a tangent vector along a long trajectory).

With the algorithm described in Sect. 4 we calculated then approximations of the attractor and the Markov measure (at least we believe that our approximation have something to do with the Markov measure) for the two regimes.

The case $\lambda_1 > 0$ is of particular interest. The randomly forced Duffing equation is (as far as we know) the first example of an SDE on $\mathbb{R}^d$, where a Markov measure with a positive Lyapunov exponent is observed. It is also possible to prove $\lambda_1 > 0$ for certain parameters analytically (Arnold and Imkeller [3]). Hence we can believe that this observation is not just a numerical artifact.

A positive Lyapunov implies the existence of (one–dimensional) unstable manifolds [31, Chapter 6], which have to be contained in the global attractor $A(\omega)$ [34, Theorem 7.3]. This implies that $A(\omega)$ has topological dimension at least one. Furthermore, $\lambda_1 > 0$ indicates sensitive dependence on initial conditions, i.e. some sort of "chaotic dynamics" on the attractor. Since the invariant measure of the Markov process is absolutely continuous, the Markov measure μ is a random SRB measure, i.e. the conditional distributions of μ_ω on the unstable manifolds are absolutely continuous and the fiber entropy is equal to $\lambda_1 > 0$ [31, Chapter 6, Theorem 1.1]. This fiber entropy [31, Chapter 1, §2] measures the complexity of the "deterministic part" of the dynamics.

Chaotic "deterministic dynamics" on a random attractor for an SDE has been observed already in numerical simulations of the attractor of the Duffing–van der Pol equation [25]. However, there the Markov measure has two negative Lyapunov exponents and its support is only a random (two–point) subset of the attractor, whereas here we have not only "transient chaos" but a "real" chaotic attractor supporting a random SRB measure.

Our numerically produced pictures (see Fig. 2.(ii)) show mass distributed over the whole attractor $A(\omega)$. Therefore we believe to "see" the SRB measure. This measure is really "random" in the sense that it is not just a small perturbation of a deterministic SRB measure.

If we produce numerical images of $A(\vartheta_{kT}\omega)$ for $k = 0, ..., n$ and some $n \in \mathbb{N}, T > 0$ (i.e. we look at the time evolution of the attractor), we observe "chaotically appearing" dynamics. Frequently "noses" are growing out of the attractor and approaching another part of it.

Summarizing we come to the conclusion that stochastic noise can turn a non–chaotic deterministic system into a highly chaotic random dynamical system. This chaotic dynamics cannot be seen by only considering the corresponding Markov process (cf. the corresponding discussion by Schimansky–Geier and Herzel [35]).

Finally let us discuss the case when $\lambda_1 < 0$. By Proposition 3 the Markov measure μ_ω must have finite support. This is affirmed by our simulations.

If the algorithm runs some time, it produces a compact set which looks like a small perturbation of the deterministic attractor. The mass is concentrated in two (random) points, which are close to the stable states of the deterministic system. The shape of this set does not change significantly under further time evolution.

However, since white noise is unbounded, there should be a small but positive probability that one of these points crosses the "potential wall" near $x = 0$. This is indeed observed if we run the algorithm some more time (see Figs. 1, 2). What we see is that not the whole attractor collapses but the mass accumulates in one (random) point (which can cross the "potential wall" from time to time under further time evolution).

Hence μ_ω seems to be a Dirac measure at a point $x(\omega)$. The distribution of the random variable $x(\cdot)$ must be the (known) invariant measure for the Markov process. The invariance of μ implies $\varphi(t, \omega)x(\omega) = x(\vartheta_t\omega)$. The random set $\{x(\cdot)\}$ is a local attractor, since it has a two–dimensional stable manifold.

It is not clear (at least to the author of the present paper) whether $\{x(\cdot)\}$ is a global attractor. The numerics suggest a negative answer, but possibly there is a very small but positive probability that the whole attractor collapses to one point.

The picture suggested by the numerics is that the attractor contains besides $x(\omega)$ a compact invariant set with "horseshoe type" chaotic dynamics. "Chaos" is observed in terms of frequently appearing "noses" which move in a similar way as in the case $\lambda_1 > 0$. This phenomenon can be explained by the unboundedness of the noise and should happen for arbitrarily small $\sigma > 0$. The existence of a "horseshoe" in the case of periodic forcing is proved via transversal intersections of the stable and unstable manifold of a hyperbolic fixed point using the Smale–Birkhoff theorem, from which a version for random dynamical systems exists [20]. Anyhow, the proof of "chaos" probably cannot be carried over to the random case, because in the case of forcing by (unbounded) white noise there is no reason for the hyperbolic fixed point to persist.

The main problem here, however, is that it is not clear to which extent we can trust the numerics. Sometimes in random dynamical systems there are dynamical phenomena which happen very rarely but which are important for the asymptotic dynamics (for instance observed by Arnold et al. [2] in simulations of the stochastic Brusselator).

There is also a problem on the theoretical side. A "collapsed" random attractor (which is typical for SDE's in dimension one, see Crauel and Flandoli [13]) reflects the "very long term" behavior, but ignores important transient dynamical features. So (at least for systems with unbounded noise) a global random attractor sometimes fails to yield a satisfactory description of the qualitative dynamics.

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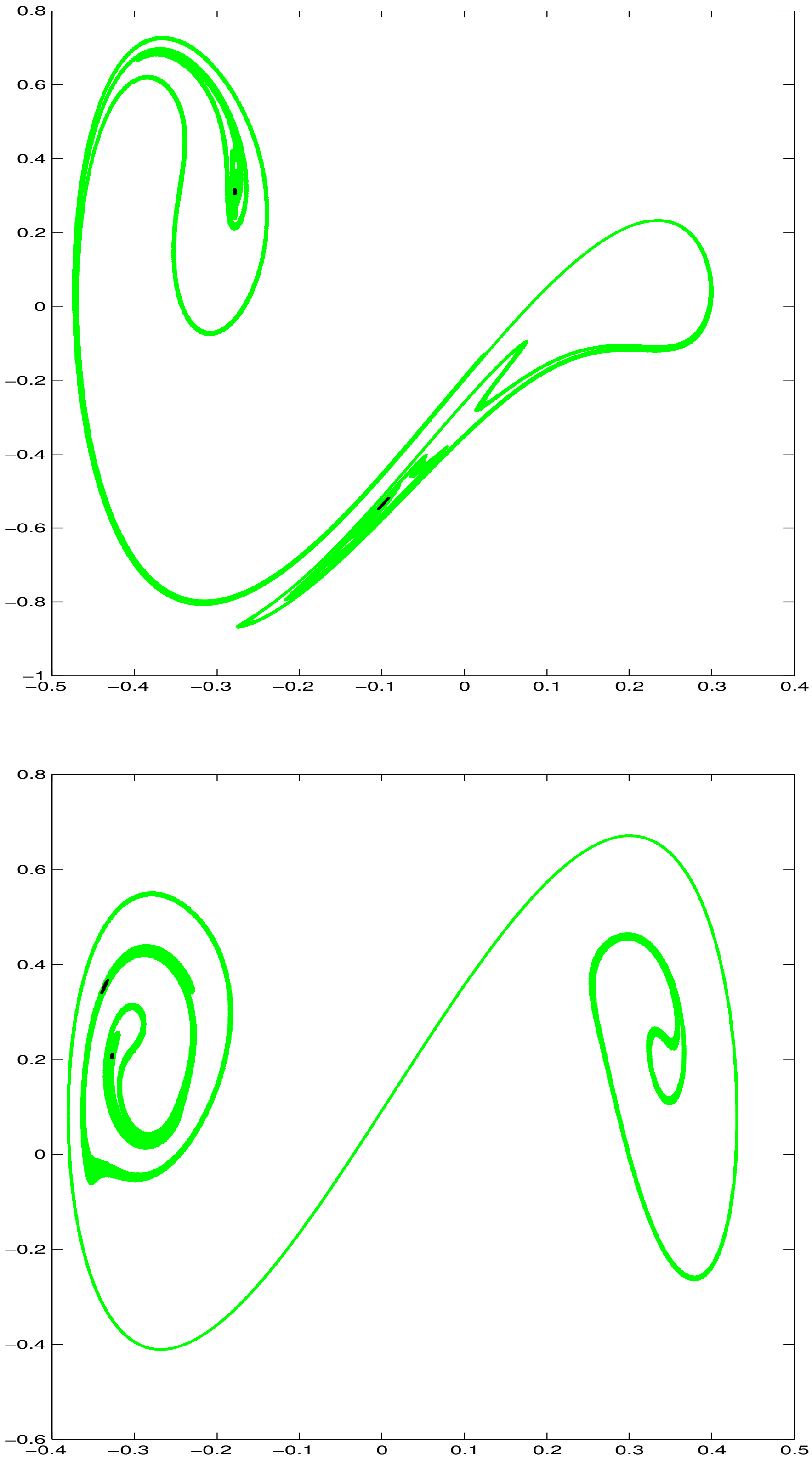

Fig. 1. Two approximations (for ω and $\theta_{2.4}\omega$) of the attractor and the Markov measure for the random Duffing oscillator in the regime $\lambda_1 < 0$. The black part has mass $1 - 10^{-6}$. The mass seems to be concentrated in two points, one of which crossing the "potential wall".

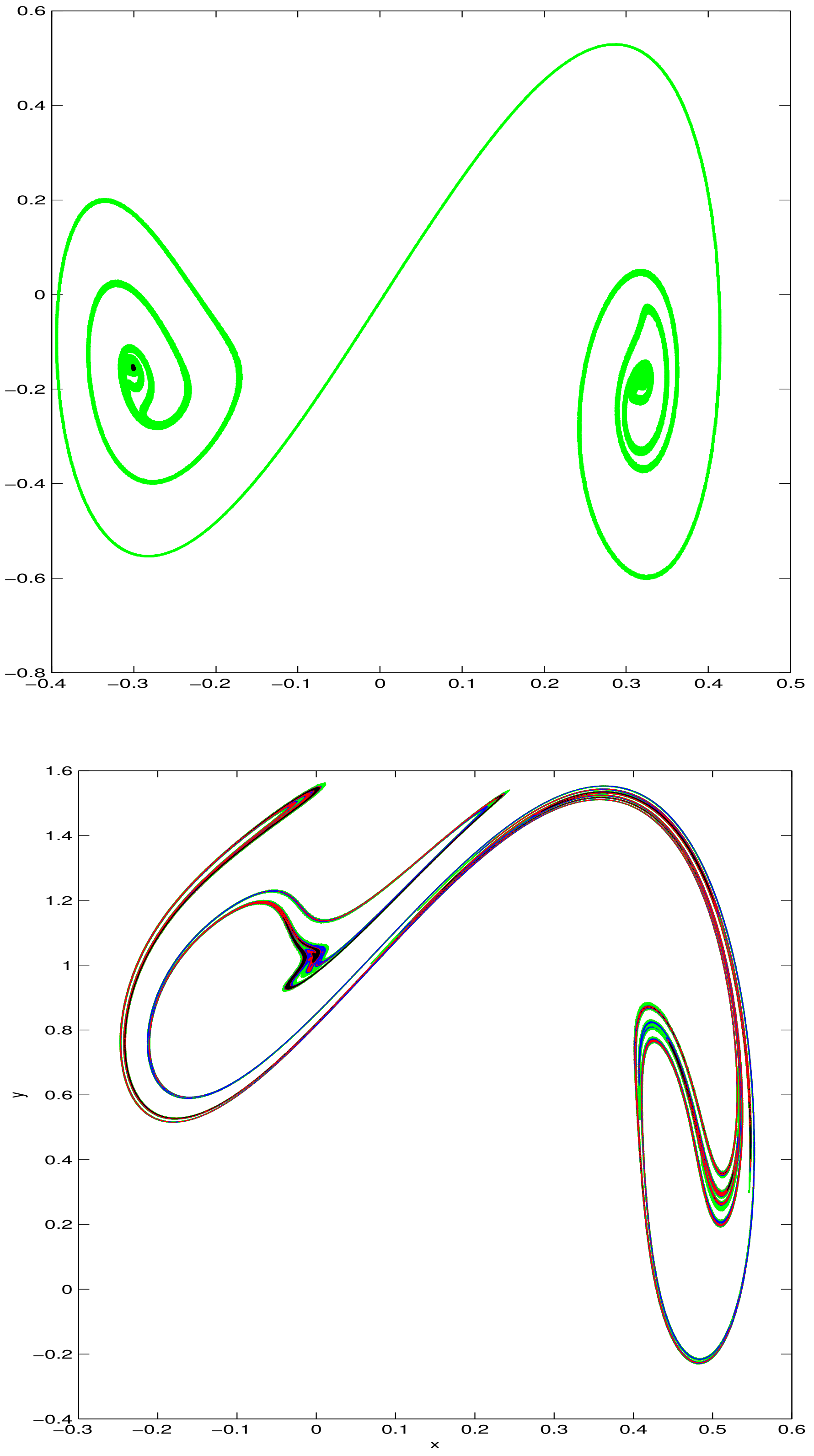

Fig. 2. (i) The final (?) picture of the random Duffing attractor in the case $\lambda_1 < 0$. The mass $(1 - 10^{-6})$ is concentrated in one (black) point.
(ii) The random Duffing attractor supporting a random SRB measure in the case $\lambda_1 > 0$. The mass distribution is indicated by different colors: 55 % green boxes with total mass 0.02, 25 % blue boxes with total mass 0.15, 16 % red boxes with total mass 0.35 and 4 % black boxes with total mass 0.5.