Evolution of microstructure: an example

Felix Otto

Institut für Angewandte Mathematik
Universität Bonn
Wegelerstr. 10
53115 Bonn, Germany

Abstract. We consider a continuum model for the flow of two phases of different mobility and density in a Hele–Shaw cell or a porous medium. As a consequence of the Saffman–Taylor instability, the phase distribution is thought to develop a microstructure, so that its evolution is effectively unpredictable.
We identify the constraints on the macroscopic quantities, like the averaged volume fraction of the phases, and show that these constraints allow to derive some predictions on how the macroscopic quantities change over time. Furthermore, we investigate a class of closure hypothesis, which complement these constraints and thereby determine an evolution of the macroscopic quantities themselves, by analyzing the stability of this evolution.
Our analysis uses a combination of tools from nonlinear scalar conservation laws and ideas from the theory of effective moduli.

1 Generation of microstructure

In this paper, we investigate a simple continuum model [12] for a saturated immiscible two-phase flow in a Hell-Shaw cell or porous medium. The two phases (indexed with h and l) have different mobilities, w.l.o.g.

$$0 \;<\; m_h \;<\; m_l,$$

and different densities. Let the open set $\Omega \subset (0,\infty)\times\mathbb{R}^N$ denote the distribution of the h-phase in space-time $(0,\infty)\times\mathbb{R}^N$ ($N=2$ for the Hell-Shaw cell and $N=3$ for the porous medium, where one should think of Ω as the mesoscopic phase distribution, as averaged over many pores). We introduce the following notation

- $\bar\Omega^c$ for the complement of the closure of Ω in $(0,\infty)\times\mathbb{R}^N$, that is, the distribution of the l-phase,
- $\partial\Omega$ for the boundary of Ω, that is, the interface,
- ν for the inner spatial normal to $\partial\Omega$,
- V for the normal velocity of $\partial\Omega$.

The velocity u, an $\mathbb{R}^N$-valued function of $(t,x)\in(0,\infty)\times\mathbb{R}^N$, satisfies the kinematic conditions

$$\nabla\cdot u \;=\; 0 \quad\text{in}\quad \Omega\cup\bar\Omega^c, \tag{1}$$

$$u \cdot \nu \text{ is continuous across } \partial\Omega \text{ and coincides with } V. \tag{2}$$

Darcy's law states that there exists a pressure p, an $\mathbb{R}$-valued function of $(t,x) \in (0,\infty) \times \mathbb{R}^N$, such that

$$u \;=\; \left\{ \begin{array}{ll} m_h\,(-\nabla p - e) & \text{in } \Omega \\ m_l\,(-\nabla p) & \text{in } \bar{\Omega}^c \end{array} \right\}, \tag{3}$$

$$p \text{ is continuous across } \partial\Omega. \tag{4}$$

Here the given vector $e \in \mathbb{R}^N$ models the effect of density difference.

We observe that (1), (2), (3) & (4) constitute an elliptic problem for p with discontinuous coefficients and right hand side, both depending on the phase distribution. If p or the normal component of u is prescribed on the boundary of the physical domain, we see that at any time t, (1), (2), (3) & (4) determine $p(t)$ – up to an additive constant in space – and thus $u(t)$, given the spatial phase distribution $\Omega(t)$ at that time. Hence (1), (2), (3) & (4) formally define an evolution of the spatial phase distribution. By evolution we mean that we can predict the spatial phase distribution Ω at a later time if we know Ω at an earlier time. As we shall see now, this is only formally the case – effectively, Ω turns out to be unpredictable.

The effective unpredictability of Ω becomes apparent by investigating the stability of a moving planar front. A moving planar front is a solution (Ω_*, u_*, p_*) of (1), (2), (3) & (4) of the form

$$\Omega_* \;=\; \{\nu_* \cdot x > V_*\,t\}, \quad \left\{ \begin{array}{l} c := u_* \cdot \nu_* \\ h := (-\nabla p_*) - (-\nabla p_*) \cdot \nu_*\, \nu_* \end{array} \right\} \;=\; const.,$$

for given $c \in \mathbb{R}$ and $\nu_*,\ h \in \mathbb{R}^N$ with $|\nu_*| = 1$ and $h \cdot \nu_* = 0$. The normal velocity V_* is determined by (2)

$$V_* \;=\; c, \tag{5}$$

and the pressure is determined by (3)

$$-p_*(t,x) \;=\; h \cdot x \;+\; (\nu_* \cdot x - V_*\,t) \left\{ \begin{array}{ll} \frac{c}{m_h} + \rho & \text{for } \nu_* \cdot x \geq V_*\,t \\ \frac{c}{m_l} & \text{for } \nu_* \cdot x \leq V_*\,t \end{array} \right\}$$

where $\rho := e \cdot \nu_*$.

By stability investigation we mean the answer to the question: Do initially nearby solutions (Ω, u, p) of (1), (2), (3) & (4) stay close to (Ω_*, u_*, p_*), or do the deviate from (Ω_*, u_*, p_*) over time? Since (1) & (3) imply that $-\nabla^2 p = 0$ in $\Omega \cup \bar{\Omega}^c$, we make the following Ansatz for a perturbation (Ω, u, p):

$$\Omega = \{\nu_* \cdot x \;>\; V_*\,t + A\,\cos\}, \tag{6}$$

$$\begin{aligned} -p(t,x) = h \cdot x \;+\; (\nu_* \cdot x - V_*\,t) & \left\{ \begin{array}{ll} \frac{c}{m_h} + \rho & \text{for } \nu_* \cdot x \;>\; V_*\,t + A\,\cos \\ \frac{c}{m_l} & \text{for } \nu_* \cdot x \;<\; V_*\,t + A\,\cos \end{array} \right\} \\ + A & \left\{ \begin{array}{ll} a_h\,\exp_h\,\cos \;+\; b_h\,\exp_h\,\sin & \text{for } \nu_* \cdot x \;>\; V_*\,t + A\,\cos \\ a_l\,\exp_l\,\cos \;+\; b_l\,\exp_l\,\sin & \text{for } \nu_* \cdot x \;<\; V_*\,t + A\,\cos \end{array} \right\} \end{aligned} \tag{7}$$

where we use the abbreviations

$$\left.\begin{array}{rl} A := A_0\, \exp\left(\dfrac{\mu_\nu\, t}{l} \right), & \\ \cos := \cos\left(\dfrac{\tau\cdot x - \mu_\tau\, t}{l} \right), & \sin := \sin\left(\dfrac{\tau\cdot x - \mu_\tau\, t}{l} \right), \\ \exp_h := \exp\left(-\dfrac{\nu_*\cdot x - V_*\, t}{l} \right), & \exp_l := \exp\left(\dfrac{\nu_*\cdot x - V_*\, t}{l} \right). \end{array}\right\} \tag{8}$$

We think of the direction τ (with $\tau\cdot\nu_* = 0$), the initial amplitude A_0 and the wavelength $2\pi l$ of the perturbation as given. As we shall see, the six other parameters (μ_ν, μ_τ, a_h, a_l, b_h and b_l) are determined by the requirement that (Ω, u, p) (with u defined as in (3)) satisfies (1), (2), (3) & (4) to first order in $A/l << 1$. The parameter of interest is the growth rate μ_ν/l of the amplitude. This linear stability analysis is standard [12,3] and is reproduced here only for the convenience of the reader and since we use a similar calculus in Chapter 4.

The two traces of the function (7) on the interface $\partial\Omega$, one coming from Ω and the other from $\bar\Omega^c$, are given by

$$\begin{aligned} -\tfrac{1}{l}\, p_{|\Omega} &\doteq \tfrac{1}{l}\, h\cdot x + \tfrac{A}{l}\left\{ \left(\tfrac{c}{m_h} + \rho\right) \cos + a_h \cos + b_h \sin \right\}, \\ -\tfrac{1}{l}\, p_{|\bar\Omega^c} &\doteq \tfrac{1}{l}\, h\cdot x + \tfrac{A}{l}\left\{ \tfrac{c}{m_l} \cos + a_l \cos + b_l \sin \right\}, \end{aligned}$$

where "$\doteq$" denotes equality up to first order in A/l. (4), that is, the continuity of p across $\partial\Omega$ therefore imposes

$$\tfrac{c}{m_h} + \rho + a_h = \tfrac{c}{m_l} + a_l \quad \text{and} \quad b_h = b_l.$$

Since the perturbed normal ν satisfies

$$\nu \doteq \nu_* + \tfrac{A}{l}\, \tau \sin,$$

we have for the two traces of the normal derivative of the function in (7)

$$\begin{aligned} -\nabla p\cdot\nu_{|\Omega} &\doteq \tfrac{c}{m_h} + \rho + \tfrac{A}{l}\left\{ h\cdot\tau \sin - a_h \cos - b_h \sin \right\}, \\ -\nabla p\cdot\nu_{|\bar\Omega^c} &\doteq \tfrac{c}{m_l} + \tfrac{A}{l}\left\{ h\cdot\tau \sin + a_l \cos + b_l \sin \right\}. \end{aligned}$$

Together with

$$-e\cdot\nu_{|\Omega} \doteq -\rho - \tfrac{A}{l}\, e\cdot\tau \sin,$$

we obtain from (3)

$$\left.\begin{aligned} u\cdot\nu_{|\Omega} &\doteq c + \tfrac{A}{l}\, m_h \left\{ h\cdot\tau \sin - a_h \cos - b_h \sin - e\cdot\tau \sin \right\}, \\ u\cdot\nu_{|\bar\Omega^c} &\doteq c + \tfrac{A}{l}\, m_l \left\{ h\cdot\tau \sin + a_l \cos + b_l \sin \right\}. \end{aligned}\right. \tag{9}$$

Hence (2), that is, the continuity of $u\cdot\nu$ across $\partial\Omega$ imposes

$$-m_h\, a_h = m_l\, a_l \quad \text{and} \quad m_h\, (h\cdot\tau - b_h - e\cdot\tau) = m_l\, (h\cdot\tau + b_l).$$

Thus we are forced to choose

$$\begin{array}{rcl} -m_h\, a_h = m_l\, a_l &=& \frac{m_l - m_h}{m_l + m_h}\, c \qquad + \ \frac{m_l\, m_h}{m_l + m_h}\, \rho, \\ -b_h \quad = \ -b_l &=& \frac{m_l - m_h}{m_l + m_h}\, h \cdot \tau \ + \ \frac{m_h}{m_l + m_h}\, e \cdot \tau. \end{array}$$

Since the normal velocity V of the boundary of (6) is to first order given by

$$V \ \doteq \ V_* \ + \ \tfrac{A}{l}\ \{\mu_\nu \cos + \mu_\tau \sin\},$$

the kinematic condition $V = u \cdot \nu$ in (2) enforces in view of (9) and (5) that

$$\mu_\nu \ = \ m_l\, a_l \quad \text{and} \quad \mu_\tau \ = \ m_l\, h \cdot \tau + m_l\, b_l.$$

Hence we obtain

$$\mu_\nu \ = \ \frac{1}{\frac{1}{m_h} + \frac{1}{m_l}} \left((\tfrac{1}{m_h} - \tfrac{1}{m_l})\, c + \rho \right),$$

so that it is the sign of

$$\lambda \ := \ (\tfrac{1}{m_h} - \tfrac{1}{m_l})\, c + \rho$$

which determines the stability ($\lambda > 0$ means instable).

We see that the more mobile phase displacing the less mobile phase ($c > 0$) and the heavier phase above the lighter phase ($\rho > 0$) creates instabilities. These are the celebrated Saffman-Taylor instabilities [12], which are seen as the origin of the numerically [2] and experimentally [4] observed "fingering" of the phase distribution. We also observe that the growth rate of the amplitude

$$\frac{\mu_\nu}{l} \ = \ \frac{1}{\frac{1}{m_h} + \frac{1}{m_l}}\, \frac{\lambda}{l}$$

is inversely proportional to the wave length $2\pi l$. This reflects the scale invariance of (1), (2), (3) & (4) under the hyperbolic scaling

$$t \ \to \ \alpha\, t, \quad x \ \to \ \alpha\, x. \tag{10}$$

Hence in the unstable case, small wave length perturbations of the phase distribution grow very fast — which we interpret as effective unpredictability.

Only the modelling of an additional physical effect, which breaks the scale invariance (10), can be mathematically regularizing in the sense that it provides a lower cut-off l_- for the band of unstable wave lengths (we think of surface tension or molecular diffusion). An upper cut-off l_+ is for instance provided by the dimensions of the physical flow domain. Unfortunately, $l_- << l_+$ in the applications of interest. Hence we expect the generation and evolution of a fine scale structure of the phase distribution. The goal of this paper is to convince the reader that despite this broad band of active modes, certain aspects of the evolution of the phase distribution can be predicted. Our starting point is the unregularized set of equations (1), (2), (3) & (4), even if the corresponding evolution problem is mathematically ill-posed. It is believed that additional physical effects, like molecular diffusion, dispersion or surface tension, act as a singular perturbation and constrain the evolving microstructure beyond the predictions of this paper. We plan to analyze these further constraints in the future — this will require new methods.

The analysis presented here is an extension of earlier work by the author contained in [9–11]. Analysis from a similar point of view has been performed by Alt [1] and Luckhaus & Plotnikov [6].

2 Containment of the microstructured zone

We continue to investigate perturbations (Ω, u, p) of the moving planar front (Ω_*, u_*, p_*) in the unstable case, that is,

$$\lambda \;=\; (\tfrac{1}{m_h} - \tfrac{1}{m_l})\, c + \rho \;>\; 0,$$

this time in the nonlinear regime. We restrict ourselves to the case where the front normal ν_*, the velocity and the gravity vector are all aligned:

$$u_* \;=\; c\,\nu_* \quad \text{and} \quad e \;=\; \rho\,\nu_*. \tag{11}$$

The linear stability analysis of Chapter 1 showed that the amplitude $A(t)$ of a perturbation (Ω, u, p) of wave length $2\pi l$ grows like

$$A(t) \;\approx\; \exp\left(\frac{1}{\frac{1}{m_h}+\frac{1}{m_l}}\frac{\lambda\, t}{l}\right) A(0) \quad \text{as long as} \quad A(t) << l.$$

Loosely speaking, we will show in Proposition 1 (see Remark 2 thereafter), that in an average sense, the growth of the amplitude is estimated as follows

$$A(t) \;\leq\; A(0) \,+\, (m_l + m_h)\,\lambda\, t \quad \text{for} \quad \lambda\, t >> l. \tag{12}$$

The interpretation of this estimate is the following: The zone where the phase distribution is characterized by a finger–like microstructure can at most grow linearly in time, with a rate $(m_l + m_h)\,\lambda$. Observe that it is the sum $\frac{1}{m_h} + \frac{1}{m_l}$ of the resistancies which enters in the linear result and the sum of the mobilities $m_l + m_h$ which appears in the nonlinear result.

In Proposition 1, we show that the deviation from (12) for any perturbation is small. In this nonlinear context, by a perturbation (Ω, u, p) of wave length $2\pi l$ we understand a solution of (1), (2), (3) & (4) which is periodic of period $2\pi l$ in the tangential variables and approaches (Ω_*, u_*, p_*) for the normal variable $\to \pm\infty$. To be more precise, it is convenient to describe the phase distribution Ω by the characteristic function

$$\chi \;:=\; \left\{ \begin{array}{ll} 1 \text{ in } & \Omega \\ 0 \text{ in } & \bar{\Omega}^c \end{array} \right\}$$

and to formulate (1), (2), (3) & (4) in the weak form

$$\partial_t \chi \,+\, \nabla\cdot(\chi\, u) = 0 \quad \text{distributionally,} \tag{13}$$

$$\nabla\cdot u = 0 \quad \text{distributionally,} \tag{14}$$

$$u = (m_h\, \chi + m_l\,(1-\chi))\,(-\nabla p - \chi\, e) \quad \text{a.e..} \tag{15}$$

By periodicity of period $2\pi l$ and asymptotic behavior like (Ω_*, u_*, p_*) we understand

$$\chi, u, p \quad \text{are periodic with period } 2\pi l \text{ in} \quad y_1 := \tau_1\cdot x, \ldots, y_{N-1} := \tau_{N-1}\cdot x,$$

$$\int_{Q\cap\{z>0\}} (1-\chi) + \int_{Q\cap\{z<0\}} \chi + \int_Q |u - c\,\nu_*|^2 \;<\; \infty \quad \text{where } z := \nu_*\cdot x$$

at any given time, where $\{\tau_1, \ldots, \tau_{N-1}, \nu_*\}$ is an orthonormal basis of $\mathbb{R}^N$ and $Q := (0, 2\pi l)^{N-1} \times \mathbb{R}$ the periodic box.

The main idea is to compare the fraction of phase h at height z and time t, that is,

$$s(t,z) \;:=\; \frac{1}{(2\pi l)^{N-1}} \int_{(0,2\pi l)^{N-1}} \chi(t,y,z)\, dy \quad \in \quad [0,1],$$

with the (entropy) solution s_* of the following Riemann problem of the scalar conservation law

$$(\partial_t + c\,\partial_z) s_* + \lambda\,\partial_z[f(s_*)] \;=\; 0 \quad \text{and} \quad s_*(t=0,z) \;=\; \left\{ \begin{array}{ll} 1 \text{ for } & z>0 \\ 0 \text{ for } & z<0 \end{array} \right\}.$$

Here the flux function f

$$f(s) \;=\; -\frac{m_h s\, m_l(1-s)}{m_h s + m_l(1-s)}$$

is strictly convex. For Riemann problems for a scalar conservation law with strictly convex flux function we refer the reader for instance to [13, Chapter 16]. There is an explicit formula for s_*, which we shall introduce now. It is convenient to pass to rescaled variables

$$\hat{t} \;=\; \lambda\, t \quad \text{and} \quad \hat{z} \;=\; z - c\,t. \tag{16}$$

Since s_* solves

$$\partial_{\hat{t}} s_* + \partial_{\hat{z}}[f(s_*)] \;=\; 0 \quad \text{and} \quad s_*(\hat{t}=0,\hat{z}) \;=\; \left\{ \begin{array}{ll} 1 \text{ for } & \hat{z}>0 \\ 0 \text{ for } & \hat{z}<0 \end{array} \right\}, \tag{17}$$

it is of self-similar form

$$s_* \;=\; s_*(\hat\zeta) \quad \text{with} \quad \hat\zeta \;=\; \hat z/\hat t,$$

where the profile is implicitly given by

$$f'(s_*(\hat\zeta)) \;=\; \left\{ \begin{array}{ccc} f'(1) & \text{for} & f'(1) \,\le\, \hat\zeta \\ \hat\zeta & \text{for} & f'(0) \,\le\, \hat\zeta \,\le\, f'(1) \\ f'(0) & \text{for} & \hat\zeta \,\le\, f'(0) \end{array} \right\}. \tag{18}$$

A straightforward but tedious calculation shows that s_* is explicitly given by

$$s_*(\hat\zeta) \;=\; \left\{ \begin{array}{cc} 1 & \text{for} \quad m_l \,\le\, \hat\zeta \\ \frac{m_h^{-1} - (m_h\, m_l + (m_l - m_h)\,\hat\zeta)^{-1/2}}{m_h^{-1} - m_l^{-1}} & \text{for} \quad -m_h \,\le\, \hat\zeta \,\le\, m_l \\ 0 & \text{for} \quad \hat\zeta \,\le\, -m_h \end{array} \right\}. \tag{19}$$

Proposition 1. *Assume that $z_{h,0} \le z_{l,0}$ are so that initially,*

$$\int_{-\infty}^{z_{h,0}} s(0,z)\, dz \;=\; 0 \qquad \textit{and} \qquad \int_{z_{l,0}}^{+\infty} (1-s)(0,z)\, dz \;=\; 0.$$

Then we have for all t with $\frac{m_h\,\lambda\, t}{l} \ge 2$ and all $\hat\zeta \in \mathbb{R}$ the two estimates

$$\frac{1}{(m_l\, m_h)^{1/2}} \left\{ \begin{array}{c} \frac{1}{\lambda\, t} \displaystyle\int_{-\infty}^{z_{h,0}+c\,t+\hat\zeta\,\lambda\, t} s\;(t,z)\, dz \;-\; \int_{-\infty}^{\hat\zeta} s_*\;(\hat\zeta')\, d\hat\zeta' \\ \frac{1}{\lambda\, t} \displaystyle\int_{z_{l,0}+c\,t+\hat\zeta\,\lambda\, t}^{+\infty} (1-s)(t,z)\, dz \;-\; \int_{\hat\zeta}^{+\infty} (1-s_*)(\hat\zeta')\, d\hat\zeta' \end{array} \right\}$$
$$\le\; C \left(\frac{l}{m_h\,\lambda\, t} \right)^{1/2} \log \frac{m_h\,\lambda\, t}{l}, \tag{20}$$

where $C < \infty$ is a universal constant.

Remark 2. In view of (19), the choices $\hat\zeta = -m_h$ and $\hat\zeta = m_l$ in Proposition 1 yield

$$\frac{1}{(m_l\, m_h)^{1/2} \lambda\, t} \left\{ \int_{-\infty}^{z_{h,0}+c\,t-m_h\,\lambda\, t} s(t,z)\, dz \;+\; \int_{z_{l,0}+c\,t+m_l\,\lambda\, t}^{+\infty} (1-s)(t,z)\, dz \right\}$$
$$\le\; C \left(\frac{l}{m_h\,\lambda\, t} \right)^{1/2} \log \frac{m_h\,\lambda\, t}{l}$$

for all t with $\frac{m_h\,\lambda\, t}{l} \ge 2$. The left hand side measures the deviation from (12), in the sense that it measures the volume of the part of the phase distribution which violates (12). It measures the normalized volume; normalized by the reference volume of $(0, 2\pi l)^{N-1} \times (-m_h\lambda t, m_l\lambda t)$. The estimate states that this normalized volume decreases as $t^{-1/2}$ as time goes to infinity and as $l^{1/2}$ as the perturbation length goes to zero.

Evoking the underlying symmetry, we restrict ourselves to the proof of the first part of Proposition 1. By translational symmetry, we may assume $z_{h,0} = 0$. We will reformulate the first part of Proposition 1 in terms of the distribution functions of s and s_*, that is

$$V(t,z) \;=\; \int_{-\infty}^{z} s(t,\tilde z)\,d\tilde z \quad \text{resp.} \quad V_*(t,z) \;=\; \int_{-\infty}^{z} s_*(t,\tilde z)\,d\tilde z$$

and the rescaled variables (16). We observe that V_* inherits the self similarity of s_*

$$V_*(\hat t,\hat z) \;=\; \hat t \int_{-\infty}^{\hat\zeta} s_*(\hat\zeta')\,d\hat\zeta \quad \text{where} \quad \hat\zeta \;=\; \hat z/\hat t, \tag{21}$$

so that

$$\frac{1}{\hat t}\left(V(\hat t,\hat z) \;-\; V_*(\hat t,\hat z)\right) \;=\; \frac{1}{\lambda t}\int_{-\infty}^{ct+\hat\zeta\lambda t} s(t,\tilde z)\,d\tilde z \;-\; \int_{-\infty}^{\hat\zeta} s_*(\hat\zeta')\,d\hat\zeta'.$$

On the other hand, we have according to $z_{0,h} = 0$

$$V(0,\hat t) - V_*(0,\hat t) \;=\; V(0,\hat t) - \hat t_+ \;\le\; 0,$$

so that the first part of Proposition 1 is a consequence of

Proposition 3. *There exists a universal constant $C < \infty$, s. t. for all $\hat t$ with $\frac{m_h\,\hat t}{l} \ge 2$*

$$\frac{1}{(m_h\,m_l)^{1/2}\,\hat t}\left\{\sup_{\hat z\in\mathbb{R}}(V(\hat t,\hat z) - V_*(\hat t,\hat z))_+ \;-\; \sup_{\hat z\in\mathbb{R}}(V(0,\hat z) - V_*(0,\hat z))_+\right\}$$
$$\le C\left(\frac{l}{m_h\,\hat t}\right)^{1/2}\log\frac{m_h\,\hat t}{l}.$$

For the proof of Proposition 3 we observe that

$$\begin{aligned} s(t,z) &= \frac{1}{(2\pi l)^{N-1}}\int_{(0,2\pi l)^{N-1}} \chi(t,y,z)\,dy,\\ c &= \frac{1}{(2\pi l)^{N-1}}\int_{(0,2\pi l)^{N-1}} u(t,y,z)\cdot\nu_*\,dy \end{aligned} \tag{22}$$

and notice that

$$j(t,z) := \frac{1}{\lambda}\,\frac{1}{(2\pi l)^{N-1}}\int\{\chi(t,y,z)\,u(t,y,z)\cdot\nu_* - s(t,z)\,c\}\,dy \tag{23}$$

is such that (13) in the rescaled variables projects onto

$$\partial_{\hat t} s + \partial_{\hat z} j \;=\; 0 \quad \text{distributionally.}$$

In view of (17), we have to relate j to $f(s)$ in order to compare s to s_*. As can be seen from (23) j has an interesting interpretation: For the quantities χ and $u\cdot\nu_*$, j measures the deviation of the horizontal average of the product from the product of the horizontal averages. If one is willing to give up complete localization in z, one obtains a fairly sharp estimate of j in terms of $f(s)$.

Lemma 4. *We have $\int_{-\infty}^{+\infty}|j|\,dz<\infty$ and if $\varphi=\varphi(z)$ is positive and slowly varying on scale L in the sense of*

$$L\left|\frac{d\varphi}{dz}\right| \;\le\; \varphi, \tag{24}$$

then we have for all $t\in(0,\infty)$

$$\Big|\int_{-\infty}^{+\infty} j\,\varphi\,dz\Big| \;\le\; \frac{1}{1-\frac{l}{L}(\frac{m_l}{m_h})^{1/2}}\int_{-\infty}^{+\infty}(-f(s))\,\varphi\,dz. \tag{25}$$

At the heart of Lemma 4 is the following elliptic estimate

Lemma 5. *Let $m,\tilde p,\tilde u$ and $\tilde f$ be periodic in y with $\nabla\tilde p,\ \tilde u,\ \tilde f\,\in\,L^2(Q)^N$. Assume that*

$$\nabla\cdot\tilde u = 0 \quad \textit{distributionally},$$

$$\tilde u = m\,(-\nabla\tilde p-\tilde f) \quad \textit{a.e.}, \tag{26}$$

$$m\in[m_h,m_l] \quad \textit{a.e.}. \tag{27}$$

Then for any φ as in Lemma 4 we have

$$\int_Q m\,|\nabla\tilde p|^2\,\varphi \;+\; \int_Q\frac{1}{m}\,|\tilde u|^2\,\varphi \;\le\; \frac{1}{1-\frac{l}{L}(\frac{m_l}{m_h})^{1/2}}\int_Q m\,|\tilde f|^2\,\varphi. \tag{28}$$

We now use the conservation laws

$$\left.\begin{array}{l}\partial_{\hat t}s \;\;+\partial_{\hat z}j \qquad\quad = 0\\ \partial_{\hat t}s_* + \partial_{\hat z}[f(s_*)] = 0\end{array}\right\} \tag{29}$$

and Lemma 4 to compare s and s_* directly.

Lemma 6. *Let $\varphi=\varphi(\hat\zeta), \hat\zeta=\hat z/\hat t$, be bounded, decreasing with derivative slowly varying on scale Λ in the sense of*

$$\Lambda\left|\frac{d^2\varphi}{d\hat\zeta^2}\right| \le -\frac{d\varphi}{d\hat\zeta}. \tag{30}$$

Let $\hat\tau=\hat z(\hat t)$ be an adapted reparametrization of time given by

$$\left\{\begin{array}{ll}\frac{d\hat\tau}{d\hat t} = 1-\frac{l}{\Lambda\hat t}(\frac{m_l}{m_h})^{1/2} & \textit{for } \hat t>\frac{l}{\Lambda}(\frac{m_l}{m_h})^{1/2}\\ \hat\tau\; = 0 & \textit{for } \hat t=\frac{l}{\Lambda}(\frac{m_l}{m_h})^{1/2}\end{array}\right\}. \tag{31}$$

Then

$$\frac{d}{d\hat t}\int_{-\infty}^{+\infty}(s\circ\hat\tau-s_*)\,\varphi\,d\hat z \;\le\; 0 \qquad \textit{for}\quad \hat t>\frac{l}{\Lambda}(\frac{m_l}{m_h})^{1/2}.$$

We now choose φ appropriately to localize the result of Lemma 6

Lemma 7.

$$\sup_{\hat z\in\mathbb{R}}\big(V(\hat\tau,\hat z)-V_*(\hat t,\hat z)\big)_+ \;-\; \sup_{\hat z\in\mathbb{R}}\Big(V(0,\hat z)-V_*\big(\tfrac{l}{\Lambda}(\tfrac{m_l}{m_h})^{1/2},\hat z\big)\Big)_+ \;\le\; \Lambda\,\hat t.$$

We now pass from $V_*(\hat t,\hat z)$ to $V_*(\hat\tau,\hat z)$ using the Lipschitz continuity of V_* in time.

Lemma 8.

$$\sup_{\hat z\in\mathbb{R}}(V-V_*)_+(\hat\tau,\hat z) \;-\; \sup_{\hat z\in\mathbb{R}}(V-V_*)_+(0,\hat z) \;\le\; \frac{1}{4}\,(m_h\,m_l)^{1/2}\,(\hat t-\hat\tau)+\Lambda\,\hat t.$$

We finally optimize the parameter Λ for a given $\hat\tau$.

Lemma 9. *. There exists a universal constant $C<\infty$ s. t. for all $\hat\tau$ with $\frac{m_h\,\hat\tau}{l}\ge 2$,*

$$\sup_{\hat z\in\mathbb{R}}(V-V_*)_+(\hat\tau,\hat z) \;-\; \sup_{\hat z\in\mathbb{R}}(V-V_*)_+(0,\hat z) \;\le\; C\,(m_l\,l\,\hat\tau)^{1/2}\,\log\frac{m_h\,\hat\tau}{l}.$$

3 Coarse graining of the equations

In Chapter 1, we saw that although (13), (14) & (15) formally define an evolution of the microscopic spatial phase distribution χ, χ is effectively unpredictable from the knowledge of χ at an earlier time. In Chapter 2, we worked out that certain aspects of macroscopic averages s of χ are predictable from the knowledge of s at an earlier time. In this chapter, we continue to investigate the constraints on the macroscopic average s of χ, but in a more general, albeit qualitative fashion.

We cannot expect to predict s from its knowledge at earlier time, since we encounter the following closure problem: Taking macroscopic averages, denoted by "-", does not commute with nonlinearities. More precisely, although the linear differential operators commute with taking macroscopic averages,

$$\left.\begin{array}{r}\partial_t\bar\chi\;+\;\nabla\cdot(\overline{\chi u})=0\\ \nabla\cdot\bar u=0\end{array}\right\}\quad\text{distributionally,}$$

the nonlinear pointwise relations are not preserved

$$\left.\begin{array}{l}\overline{\chi u}\neq\bar\chi\,\bar u\\ \;\bar u\;\neq(m_h\bar\chi+m_l(1-\bar\chi))\,(-\nabla\bar p-\bar\chi\,e)\end{array}\right\}\quad\text{a. e. .}\qquad(32)$$

The first defect in (32) has been estimated in Lemma 4 of Chapter 2. In Proposition 11 we point out that the only object required to describe both defects in (32) is the effective mobility M of the microscopic mobility $m_h\chi + m_l(1-\chi)$.

Loosely speaking, the effective mobility M is characterized by the following property: whenever

$$\nabla\cdot v \;=\; 0 \quad \text{and} \quad v \;=\; (m_h\chi + m_l(1-\chi))(-\nabla q),$$

then

$$\nabla\cdot\bar{v} \;=\; 0 \quad \text{and} \quad \bar{v} \;=\; M\,(-\nabla\bar{q}).$$

M is a symmetric $N\times N$-matrix (an element of $\mathrm{Sym}(\mathbb{R}^N)$) with

$$(\tfrac{1}{m_h}\bar{\chi} + \tfrac{1}{m_l}(1-\bar{\chi}))^{-1}\,\mathrm{id} \;\le\; M \;\le\; (m_h\bar{\chi} + m_l(1-\bar{\chi}))\,\mathrm{id} \tag{33}$$

in the sense of ordering between symmetric matrices. It is well-known that M need not to be isotropic (that is, a multiple of the identity matrix id).

After this motivation, we take a more systematic approach: We embed the space of solutions (χ,u,p) of (13), (14) & (15) into a set of solutions (s,u,p,M) of a new collection of linear differential equations and nonlinear pointwise relations which are stable under the operation of taking macroscopic averages (for s,u and p) and passing to effective moduli (for M). Of course, the smaller this set of solutions (s,u,p,M), the better. We propose to consider the set of all (s,u,p,M), where s is a $[0,1]$-valued, u an $\mathbb{R}^N$-valued, p an $\mathbb{R}$-valued and M a $\mathrm{Sym}(\mathbb{R}^N)$-valued function of (t,x), which satisfy

$$\partial_t s \;+\; \nabla\cdot(G\,u + F\,e) \;=\; 0 \quad \text{distributionally,} \tag{34}$$

$$\nabla\cdot u \;=\; 0 \quad \text{distributionally,} \tag{35}$$

$$u \;=\; M\,(-\nabla p - G\,e) \quad \text{a. e. ,} \tag{36}$$

$$(\tfrac{1}{m_h}s + \tfrac{1}{m_l}(1-s))^{-1}\,\mathrm{id} \;\le M \le\; (m_h s + m_l(1-s))\,\mathrm{id} \quad \text{a. e. ,} \tag{37}$$

where the $\mathrm{Sym}(\mathbb{R}^N)$-valued G and F are given by

$$G \;:=\; \frac{M^{-1} - m_l^{-1}\,\mathrm{id}}{m_h^{-1} - m_l^{-1}}, \qquad F \;:=\; \frac{G - s\,\mathrm{id}}{m_h^{-1} - m_l^{-1}}. \tag{38}$$

The mathematically convenient way of expressing stability of (34), (35), (36) & (37) under the operation of taking averages (for s,u and p) and passing to the effective moduli (for M) in a qualitative sense is the following: The set of solutions (s,u,p,M) of (34), (35), (36) & (37) is closed under weak convergence (for s,u and p) and H-convergence (for M). For the convenience of the reader, we recall the definition of H-convergence here — in particular since we need a slightly modified notion due to the time dependence of the fields. For the unmodified notion, we refer the reader to Murat & Tartar [8] and [14].

Definition 10. A sequence $\{M_n\}_{n\uparrow\infty}$ of $\mathrm{Sym}(\mathbb{R}^N)$–valued functions of (t,x) is said to H-converge to a $\mathrm{Sym}(\mathbb{R}^N)$-valued M if the following holds: For any $\mathbb{R}^N$–valued v_n, v, and $\mathbb{R}$–valued q_n, q with

$$\begin{aligned} v_n \rightharpoonup v \quad \text{weakly in } L^2, &\qquad \nabla q_n \rightharpoonup \nabla q \quad \text{weakly in } L^2,\\ \nabla\cdot v_n \;=\; \nabla\cdot v \quad \text{distributionally}, \end{aligned}$$

we have

$$v_n \;=\; M_n\,(-\nabla q_n) \quad \text{a.e.} \quad \Longrightarrow \quad v \;=\; M\,(-\nabla q) \quad \text{a.e..}$$

Proposition 11. *i) (34), (35), (36) & (37) turns into (13), (14) & (15) for* $s=\chi\in\{0,1\}$ *since*

$$\left.\begin{aligned} \frac{m_h\,s}{m_h\,s+m_l\,(1-s)}\,\mathrm{id}\le G\le s\,\mathrm{id}\\ -\frac{m_h\,s\,m_l\,(1-s)}{m_h\,s+m_l\,(1-s)}\,\mathrm{id}\le F\le 0 \end{aligned}\right\}\quad a.e.\ . \tag{39}$$

ii) (34), (35), (36) & (37) is closed under the operation of taking macroscopic averages (for s,u,p*) and passing to effective moduli (for* M*) in the following sense: If the elements of the sequence* $\{(s_n,u_n,p_n,M_n)\}_{n\uparrow\infty}$ *satisfy (34), (35), (36) & (37), and the sequence converges to* (s,u,p,M) *in the sense of*

$$\left.\begin{array}{rl} s_n\rightharpoonup s & \textit{weak–}*\textit{ convergence in } L^\infty,\\ (u_n,\nabla p_n)\rightharpoonup(u,\nabla p) & \textit{weak convergence in } L^2,\\ M_n\to M & H-\textit{convergence in the sense of Definition 10} \end{array}\right\} \tag{40}$$

then (s,u,p,M) *satisfies (34), (35), (36) & (37).*

Remark 12. Besides mimicking the operations of taking the macroscopic average and passing to the effective modulus, the convergences (40) are also natural in the following sense: The set of solutions (s,u,p,M) of (34), (35), (36) & (37) is not only closed, but also "almost" compact in this topology. Indeed, according to $s_n\in[0,1]$ a.e., $\{s_n\}_{n\uparrow\infty}$ is bounded in L^∞. Controlled boundary conditions on the pressure automatically yield boundedness of $\{u_n\}_{n\uparrow\infty}$ and $\{\nabla p_n\}_{n\uparrow\infty}$ in L^2, according to the elliptic equation (35) & (36) with controlled coefficients (37). If it weren't for the time-dependence of M_n, the control (37) would ensure the compactness of $\{M_n\}_{n\uparrow\infty}$ w.r.t. H-convergence [8, Theorem 2].

PROOF OF PROPOSITION 11. The upper bounds $G\le s\,\mathrm{id}$ and $F\le 0$ are immediate from (37) in the form of

$$M^{-1}\;\le\;\left(\tfrac{1}{m_h}s+\tfrac{1}{m_l}(1-s)\right)\mathrm{id}$$

and the definitions (38) if G and F. The lower bounds in (39) follow from (37) in the form of

$$M^{-1}\;\ge\;\tfrac{1}{m_h s+m_l(1-s)}\,\mathrm{id}$$

and the identities

$$\begin{aligned}
\frac{1}{\frac{1}{m_h}-\frac{1}{m_l}}\left(\frac{1}{m_h s+m_l(1-s)}-\frac{1}{m_l}\right) &= \frac{m_h s}{m_h s+m_l(1-s)},\\
\frac{1}{\frac{1}{m_h}-\frac{1}{m_l}}\left(\frac{m_h s}{m_h s+m_l(1-s)}-s\right) &= -\frac{m_h s\, m_l(1-s)}{m_h s+m_l(1-s)}.
\end{aligned}$$

We now address part ii). The fact that (34), (35), (36) are preserved in the limit follows from two algebraic observations: (36) is preserved since it can be rewritten as

$$u+\frac{1}{\frac{1}{m_h}-\frac{1}{m_l}}\,e \;=\; M\left(-\nabla p+\frac{1}{m_l}\,\frac{1}{\frac{1}{m_h}-\frac{1}{m_l}}\,e\right), \tag{41}$$

which is an immediate consequence of the definition of G in (38). By definition of H-convergence, (41) is stable, since $u+\frac{1}{\frac{1}{m_h}-\frac{1}{m_l}}\,e$ is divergence–free and $\nabla p-\frac{1}{m_l}\frac{1}{\frac{1}{m_h}-\frac{1}{m_l}}\,e$ is a gradient. (34) is preserved since the flux can be rewritten as

$$\begin{aligned}
&G\,u+F\,e\\
&\overset{(38)}{=} G\,(u+\tfrac{1}{\frac{1}{m_h}-\frac{1}{m_l}}\,e)\;-\;\tfrac{1}{\frac{1}{m_h}-\frac{1}{m_l}}\,s\,e\\
&\overset{(38)}{=} \tfrac{1}{\frac{1}{m_h}-\frac{1}{m_l}}\,M^{-1}\,(u+\tfrac{1}{\frac{1}{m_h}-\frac{1}{m_l}}\,e)\;-\;\tfrac{1}{m_l}\,\tfrac{1}{\frac{1}{m_h}-\frac{1}{m_l}}\,(u+\tfrac{1}{\frac{1}{m_h}-\frac{1}{m_l}}\,e)\;-\;\tfrac{1}{\frac{1}{m_h}-\frac{1}{m_l}}\,s\,e\\
&\overset{(41)}{=} \tfrac{1}{\frac{1}{m_h}-\frac{1}{m_l}}\,(-\nabla p+\tfrac{1}{m_l}\,\tfrac{1}{\frac{1}{m_h}-\frac{1}{m_l}}\,e)\;-\;\tfrac{1}{m_l}\,\tfrac{1}{\frac{1}{m_h}-\frac{1}{m_l}}\,(u+\tfrac{1}{\frac{1}{m_h}-\frac{1}{m_l}}\,e)\;-\;\tfrac{1}{\frac{1}{m_h}-\frac{1}{m_l}}\,s\,e,
\end{aligned}$$

which is a linear expression in (s,u,p).

The argument that the inequalities (37) are preserved under weak convergence in s and H-convergence in M is only a slight modification of the one given in [14, Theorem 4 & 5].

4 A closure for the coarse grained equations

We return to the closure problem: (34), (35), (36) & (37) defines an evolution for s only if one adds a relation which allows to recover M from s, locally in time. In this chapter and the next, we will investigate the simplest form of this closure, namely

$$M \;=\; m(s)\,\mathrm{id}, \tag{42}$$

where m is a function of $s\in[0,1]$ which in view of (37) necessarily satisfies

$$\left(\tfrac{1}{m_h}\,s+\tfrac{1}{m_l}\,(1-s)\right)^{-1}\;\le\;m(s)\;\le\;m_h\,s+m_l\,(1-s)\quad\text{for all } s\in[0,1].$$

Then (34), (35) & (36) turn into

$$\partial_t s\;+\;\nabla\cdot(g(s)\,u+f(s)\,e)=0\quad\text{distributionally,} \tag{43}$$

$$\nabla\cdot u=0\quad\text{distributionally,} \tag{44}$$

$$u=m(s)\,(-\nabla p-g(s)\,e)\quad\text{a.e.,} \tag{45}$$

where

$$g(s) \;=\; \frac{(\frac{1}{m})(s)-\frac{1}{m_l}}{\frac{1}{m_h}-\frac{1}{m_l}} \quad \text{and} \quad f(s) \;=\; \frac{g(s)-s}{\frac{1}{m_h}-\frac{1}{m_l}}. \tag{46}$$

One important remark w.r.t. assumption (42) is in order: It would seem that (42) is based on the implicit assumption that the microscopic mobility $m_h\,\chi+m_l\,(1-\chi)$ is such that the effective mobility M is isotropic. In fact, (42) only implies that the macroscopic quantity $(\frac{1}{m_h}-\frac{1}{m_l})\,u+e$ is an eigenvector of the effective mobility M with eigenvalue depending only on the macroscopic variable s, as can be seen from (41) in the proof of Proposition 11.

In this chapter, we derive an important consistency property for the class of models (42): In Proposition 15, we show that longitudinal and transversal stability of a moving planar front coincide for this class of models, provided $\frac{1}{m}$ is strictly convex.

By a moving planar front we understand a solution (s_*,u_*,p_*) of (43), (44) & (45) of the form

$$\left.\begin{array}{c} s_* \;=\; \left\{\begin{array}{ll} s_h & \text{for } x\cdot\nu_* > V_*\,t \\ s_l & \text{for } x\cdot\nu_* < V_*\,t \end{array}\right\} \\ \left\{\begin{array}{lcl} c &:=& u_*\cdot\nu_* \\ h &:=& (-\nabla p_*)-(-\nabla p_*)\cdot\nu_*\,\nu_* \end{array}\right\} \;=\; const., \end{array}\right\} \tag{47}$$

where $0\le s_l<s_h\le 1$, $c\in\mathbb{R}$ and ν_*, $h\in\mathbb{R}^N$ with $|\nu_*|=1$ and $\nu_*\cdot h=0$ are given. The normal velocity V_* ist then determined by (43):

$$V_* \;=\; \frac{(c\,g+\rho\,f)(s_h)-(c\,g+\rho\,f)(s_l)}{s_h-s_l}, \tag{48}$$

where $\rho=e\cdot\nu_*$. The pressure p_* is determined by (44) & (45):

$$-p_*(t,x) \;=\; h\cdot x \;+\; (\nu_*\cdot x-V_*\,t)\left\{\begin{array}{ll} \frac{c}{m(s_h)}+\rho\,g(s_h) & \text{for } x\cdot\nu_*>V_*\,t \\ \frac{c}{m(s_l)}+\rho\,g(s_l) & \text{for } x\cdot\nu_*<V_*\,t \end{array}\right\}.$$

We claim that longitudinal and transversal stability of s_* coincide.

By stability we mean that initial perturbations do not grow over time. By longitudinal stability, we mean that (s_*,u_*,p_*) is stable under all one-dimensional perturbations (s,u,p), that is, solutions of (43), (44) & (45) of the form

$$s \;=\; s(t,x\cdot\nu_*),\quad u\cdot\nu_* \;=\; c \quad \text{and} \quad -\nabla p-(-\nabla p)\cdot\nu_*\,\nu_* \;=\; h.$$

Then s is a weak solution of the scalar conservation law

$$\partial_t s \;+\; \partial_z(c\,g(s)+\rho\,f(s)) \;=\; 0, \tag{49}$$

It is well–known that the special weak solution s_* of (49), a “shock”, is stable under the formation of “rarefaction waves”, also solutions s of (49), if and only if the following condition is fulfilled

$$(c\,g+\rho\,f)(s) \;\ge\; (c\,g+\rho\,f)(s_l)+V_*\,(s-s_l) \quad \text{for all } s\in[s_l,s_h], \tag{50}$$

see for instance [7, Chapter 2, Example (3.11) and (3.13)] and [13, Chapter 15, Example 6] for this admissibility criterion for shocks. In view of (48) and (50) we say

Definition 13. (s_*, u_*, p_*) is longitudinally stable if the following "chord"–condition is fulfilled

$$(c\,g+\rho\,f)(s) \;\geq\; \tfrac{s-s_l}{s_h-s_l}\,(c\,g+\rho\,f)(s_h) + \tfrac{s_h-s}{s_h-s_l}\,(c\,g+\rho\,f)(s_l) \quad \text{for all } s\in[s_l,s_h]. \tag{51}$$

By transversal stability we mean that (s_*, u_*, p_*) is stable under all perturbations (s, u, p) with $s \in \{s_l, s_h\}$. In fact we mean linear stability as in Chapter 1. As there, we make the following Ansatz for the perturbation (s, u, p):

$$\begin{aligned}
s(t,x) &= \left\{\begin{array}{ll} s_h & \text{for } \nu_*\cdot x > V_*\,t + A\,\cos \\ s_l & \text{for } \nu_*\cdot x < V_*\,t + A\,\cos, \end{array}\right\} \\
-p(t,x) &= h\cdot x \\
&\quad + (\nu_*\cdot x - V_*\,t)\left\{\begin{array}{lll} \frac{c}{m(s_h)} + \rho\,g(s_h) & \text{for} & \nu_*\cdot x \;>\; V_*\,t + A\,\cos \\ \frac{c}{m(s_l)} + \rho\,g(s_l) & \text{for} & \nu_*\cdot x \;<\; V_*\,t + A\,\cos \end{array}\right\} \\
&\quad + A\left\{\begin{array}{lll} a_h\,\exp_h\,\cos \;+\; b_h\,\exp_h\,\sin & \text{for} & \nu_*\cdot x \;>\; V_*\,t + A\,\cos \\ a_l\,\exp_l\,\cos \;+\; b_l\,\exp_l\,\sin & \text{for} & \nu_*\cdot x \;<\; V_*\,t + A\,\cos, \end{array}\right\},
\end{aligned}$$

where we use the same abbrevations $A, \cos, \sin, \exp_h, \exp_l$ as in Chapter 1, see (8). As in Chapter 1, we consider the direction τ (with $\tau\cdot\nu_* = 0$), the initial amplitude A_0 and the wavelength $2\pi l$ of the perturbation as given and determine the remaining six parameters μ_ν, μ_τ, a_h, a_l, b_h, b_l such that (s, u, p) (with u defined to satisfy (45)) satisfies (43), (44) & (45) to first order in $A/l << 1$. As in Chapter 1, we see that continuity of p and $\nabla u\cdot\nu$ across the interface $\{\nu_*\cdot x = V_*\,t + A\,\cos\}$ imposes

$$\left.\begin{array}{rcccl} -m(s_h)\,a_h = m(s_l)\,a_l &=& \frac{m(s_l)-m(s_h)}{m(s_l)+m(s_h)}\,c &+& \frac{m(s_h)\,m(s_l)\,(g(s_h)-g(s_l))}{m(s_l)+m(s_h)}\,\rho, \\ -b_h \quad = \quad -b_l &=& \frac{m(s_l)-m(s_h)}{m(s_l)+m(s_h)}\,h\cdot\tau &+& \frac{m(s_h)\,g(s_h)-m(s_l)\,g(s_l)}{m(s_l)+m(s_h)}\,e\cdot\tau. \end{array}\right\} \tag{52}$$

The normal velocity of the interface $\{\nu_*\cdot x = V_*\,t + A\,\cos\}$ is

$$V \;\doteq\; V_* \;+\; \frac{A}{l}\;\{\mu_\nu\cos + \mu_\tau\sin\}$$

and should according to (43) be given by

$$\begin{aligned}
V \;&=\; \frac{(g(s_h)\,u\cdot\nu + f(s_h)\,e\cdot\nu) - (g(s_l)\,u\cdot\nu + f(s_l)\,e\cdot\nu)}{s_h - s_l} \\
&\overset{(48)}{=} V_* \;+\; \frac{g(s_h)-g(s_l)}{s_h-s_l}\,(u\cdot\nu - c) \;+\; \frac{f(s_h)-f(s_l)}{s_h-s_l}\,(e\cdot\nu - \rho).
\end{aligned}$$

This enforces

$$\mu_\nu \;=\; \frac{g(s_h)-g(s_l)}{s_h-s_l}\, m(s_l)\, a_l, \tag{53}$$

and a more complicated formula for μ_τ. In view of (52) and (53) we say

Definition 14. (s_*, u_*, p_*) is transversally stable if

$$\frac{1}{(\frac{1}{m})(s_h)+(\frac{1}{m})(s_l)}\,\frac{g(s_h)-g(s_l)}{s_h-s_l}\,\left((c\,\tfrac{1}{m}+\rho\, g)(s_h)-(c\,\tfrac{1}{m}+\rho\, g)(s_l)\right)\;\le\;0. \tag{54}$$

Observe that Definition 13 involves g and f, but not m, whereas Definition 14 involves m and g, but not f. The relations (46) are just such that we have

Proposition 15. *Provided $\frac{1}{m}$ is a strictly convex function of s, longitudinal stability is equivalent to transversal stability in the sense that (51) and (54) are equivalent to the same condition*

$$\lambda \;=\; (\tfrac{1}{m_h}-\tfrac{1}{m_l})\, c+\rho \;\le\; 0. \tag{55}$$

Remark 16. . It is well–known (see for instance [7, Chapter 2, (6.18), (6.19) and Remark 5.7]), that the weak formulation together with the shock admissibility criterion (50) for the scalar conservation law (49) are equivalent to its “entropy formulation”:

$$\partial_t\eta(s)\;+\;\partial_z(c\, g_\eta(s)+\rho\, f_\eta(s))\;\le\;0 \quad \text{distributionally} \tag{56}$$

for all convex functions η (“entropies”) and g_η, f_η (“entropy fluxes”) related by

$$g_\eta' \;=\; \eta'\, g' \qquad \text{and} \qquad f_\eta' \;=\; \eta'\, f'. \tag{57}$$

Kruzkov [5] has shown that the entropy formulation renders the initial value problem of a scalar conservation law well–posed. In view of this and Proposition 15, it is natural to pass from (43), (44) & (45) to

$$\partial_t\eta(s)\;+\;\nabla\cdot(g_\eta(s)\, u+f_\eta(s)\, e)\le 0 \quad \text{distributionally,} \tag{58}$$

$$\nabla\cdot u=0 \quad \text{distributionally,} \tag{59}$$

$$u=m(s)\,(-\nabla p-g(s)\, e) \quad \text{a. e. ,} \tag{60}$$

for all convex η and g_η, f_η with (57). We stress that this entropy condition is not motivated by “viscosity” (which would have no physical meaning in our coarse grained setting), but by stability considerations!

Proof of Proposition 15. We just use the definitions (46) to show that (51) and (54) are equivalent to (55). Indeed, we infer from the definition of g (46) that μ_ν can be rewritten as

$$\mu_\nu \;=\; \frac{1}{(\frac{1}{m})(s_h)+(\frac{1}{m})(s_l)}\left(\frac{(\frac{1}{m})(s_h)-(\frac{1}{m})(s_l)}{\frac{1}{m_h}-\frac{1}{m_l}}\right)^2\frac{1}{s_h-s_l}\left((\tfrac{1}{m_h}-\tfrac{1}{m_l})\, c+\rho\right),$$

so that, since $s_h > s_l$, we have $\mu_\nu \le 0$ if and only if (55) holds. On the other hand, (50) can be written as

$$h(s) \ \ge\ \tfrac{s-s_l}{s_h-s_l}\, h(s_h) \ +\ \tfrac{s_h-s}{s_h-s_l}\, h(s_l) \quad \text{for all } s \in [s_l, s_h], \tag{61}$$

where according to the definition of g and f in (46)

$$\begin{aligned} h(s) &:= c\, g(s) + \rho\, f(s) \\ &= \frac{1}{(\frac{1}{m_h} - \frac{1}{m_l})^2} \left((\tfrac{1}{m_h} - \tfrac{1}{m_l})\, c + \rho \right) (\frac{1}{m})(s) \ +\ \text{linear term in } s. \end{aligned}$$

Since by assumption, $\frac{1}{m}$ is strictly convex and $s_h > s_l$, (61) holds if and only (55) is true.

5 Stability of the coarse grained model

In this chapter, we would like to convince the reader that within the class of closures considered in Chapter 4, the model

$$m(s) \ =\ m_h\, s + m_l\, (1-s) \tag{62}$$

enjoys good stability properties. We observe that in this model f and g have the form already familiar from Chapter 2

$$\left. \begin{aligned} g(s) &= \frac{(\frac{1}{m})(s) - \frac{1}{m_l}}{\frac{1}{m_h} - \frac{1}{m_l}} = \frac{m_h\, s}{m_h\, s + m_l\,(1-s)} \\ f(s) &= \frac{g(s) - s}{\frac{1}{m_h} - \frac{1}{m_l}} = \frac{m_h\, s\, m_l\,(1-s)}{m_h\, s + m_l\,(1-s)} \end{aligned} \right\}. \tag{63}$$

An important remark with respect to the assumption (62) is in order: As for (42), (62) implies that the macroscopic quantity $(\frac{1}{m_h} - \frac{1}{m_l})\, u + e$ is an eigenvalue of the effective mobility M with eigenvalue given this time given by the average $m_h s + m_l (1-s)$ of the microscopic mobility. Loosely speaking, this is equivalent to the assumption that the microscopic mobility $m_h \chi + m_l (1-\chi)$ can only oscillate perpendicular to $(\frac{1}{m_h} - \frac{1}{m_l})\, u + e$. Hence the choice of the specific closure (62) amounts to the assumption that the microscopic phase distribution organizes itself such that it is mostly aligned with $(\frac{1}{m_h} - \frac{1}{m_l})\, u + e$. As shown in Chapter 1, the only layering of the microscopic phase distribution which is not (linearly) unstable is indeed a layering parallel to $(\frac{1}{m_h} - \frac{1}{m_l})\, u + e$. Hence the assumption is at least not unplausible.

Our argument in favor of (62) is that all one–dimensional solutions of (58), (59) & (60) are stable. We recall that by a one–dimensional solution we understand a solution (s_*, u_*, p_*) of the form

$$s_* \ =\ s_*(t, \nu_* \cdot x) \quad \text{and} \quad \left\{ \begin{array}{l} c := u_* \cdot \nu_* \\ h := (-\nabla p_*) - (-\nabla p_*) \cdot \nu_*\, \nu_* \end{array} \right\} \ =\ const.$$

for some $c \in \mathbb{R}$, $\nu_*, h \in \mathbb{R}^N$ with $|\nu_*| = 1$ and $h \cdot \nu_* = 0$. Then (58) forces s_* to be an entropy solution of the scalar conservation law

$$\partial_t s_* + \partial_z [h(s_*)] \;=\; 0, \tag{64}$$

where $z := \nu_* \cdot x$ and the flux function h is given by

$$\begin{aligned} h(s) \;&:=\; (g(s)\,u_* + f(s)\,e) \cdot \nu_* \\ &=\; c\,g(s) + \rho\,f(s) \\ &\overset{(63)}{=} \left\{ \begin{array}{l} c\,s + \lambda\,f(s) \\ \frac{\lambda}{(\frac{1}{m_l} - \frac{1}{m_h})^2} \left((\frac{1}{m})(s) - \frac{1}{m_l} \right) - \frac{\rho}{\frac{1}{m_h} - \frac{1}{m_l}}\,s \end{array} \right\} \end{aligned} \tag{65}$$

with $\rho = \nu_* \cdot e$. Furthermore, (59) & (60) force p_* to be of the form

$$-p_* \;=\; h \cdot x + \int_0^z [c\,(\tfrac{1}{m})(s_*) + \rho\,g(s_*)]. \tag{66}$$

We restrict ourselves to initial data of the form

$$s_*(t = 0, z) \;=\; \left\{ \begin{array}{ll} s_h & \text{for} \;\; z > 0 \\ s_l & \text{for} \;\; z < 0 \end{array} \right\} \quad \text{with} \quad 0 \;\le\; s_l \;<\; s_h \;\le\; 1.$$

Then s_* is uniquely determined and must be of selfsimilar form

$$s_* \;=\; s_*(\zeta) \quad \text{where} \quad \zeta \;=\; z/t.$$

In case of

$$\lambda \;=\; (\tfrac{1}{m_l} - \tfrac{1}{m_h})\,c + \rho \;\le\; 0,$$

the entropy solution of (64) is a shock

$$s_* \;=\; \left\{ \begin{array}{ll} s_h & \text{for} \;\; \zeta > 0 \\ s_l & \text{for} \;\; \zeta < 0 \end{array} \right\}$$

and we have already established in Proposition 15 that this (s_*, u_*, p_*) is nonlinearly stable under longitudinal perturbations and linearly stable under transversal perturbations (observe that $\frac{1}{m}$ is indeed strictly convex for our model (62)). In case of

$$\lambda \;=\; (\tfrac{1}{m_l} - \tfrac{1}{m_h})\,c + \rho \;>\; 0,$$

the entropy solution of (64) is a rarefaction wave implicitly given by

$$h'(s_*(\zeta)) \;=\; \left\{ \begin{array}{lll} h'(s_h) & \text{for} & h'(s_h) \;\le\; \zeta \\ \zeta & \text{for} & h'(s_l) \;\le\; \zeta \;\le\; h'(s_h) \\ h'(s_l) & \text{for} & \zeta \;\le\; h'(s_l) \end{array} \right\} \tag{67}$$

and we will show in Proposition 17 that this (s_*, u_*, p_*) is nonlinearly stable under all perturbations. We restrict ourselves to the geometry of Chapter

2, that is the alignment of velocity, gravity and front normal (11) and to $s_h = 1, s_l = 0$. As in Chapter 2, perturbations (s, u, p) of (s_*, u_*, p_*) admitted in Proposition 17 are solutions of (58), (59) & (60), which are periodic in the tangential variables and approach (s_*, u_*, p_*) for normal variable $\to \pm\infty$. More precisely, ν_* can be extended to an orthonormal basis $\{\tau_1, \ldots, \tau_{N-1}, \nu_*\}$ such that

$$s, u, p \quad \text{are periodic with period } 2\pi l \text{ in} \quad y_i \;:=\; \tau_i \cdot x, \tag{68}$$

$$\int_Q |s - s_*|^2 + \int_Q |u - u_*|^2 \;<\; \infty, \tag{69}$$

for some $l > 0$ (which turns out to be irrelevant for our result), where $Q := (0, 2\pi l)^{N-1} \times \mathbb{R}$ is the periodic box.

Proposition 17.

$$\begin{aligned} &\frac{d}{dt}\int_Q \Big\{\lambda\, m(s)\Big|\frac{(\frac{1}{m})(s) - (\frac{1}{m})(s_*)}{\frac{1}{m_h} - \frac{1}{m_l}}\Big|^2 \\ &+ (\zeta - h'(1))_+\,(1-s) \;+\; (h'(0) - \zeta)_+\, s\Big\} \;\le\; 0. \end{aligned}$$

Remark 18. Proposition 17 identifies an adapted distance between s and s_* which decreases over time. This distance dominates the $L^2(Q)$-distance between s and s_*. The distance does not depend on the period $2\pi l$.

Proof of Proposition 17. We fix a ζ_* and use the special Kruzkov entropy / entropy flux pairs, that is

$$\eta(s) \;=\; (s - s_*(\zeta_*))_+, \quad \left\{\begin{array}{l} g_\eta(s) = (g(s) - g(s_*(\zeta_*)))_+, \\ f_\eta(s) = (f(s) - f(s_*(\zeta_*)))_+, \end{array}\right\}$$

with the parameter choosen to be $s_*(\zeta_*)$; see for instance [7, Chapter 2, Remark 5.7]. The entropy inequality (58) turns into

$$\partial_t(s - s_*(\zeta_*))_+ + \nabla\cdot[(g(s) - g(s_*(\zeta_*)))_+\, u + (f(s) - f(s_*(\zeta_*)))_+\, e] \;\le\; 0 \tag{70}$$

in the distributional sense. According to the periodicity (68) and the behavior at infinity (69) we obtain from (70) by integrating over $\{z < \zeta_*\, t\} \cap Q$

$$\begin{aligned} &\frac{d}{dt}\int_{\{z<\zeta_* t\}\cap Q} (s - s_*(\zeta_*))_+\, d\mathcal{L}^N \\ &= \zeta_* \int_{\{z=\zeta_* t\}\cap Q} (s - s_*(\zeta_*))_+\, d\mathcal{H}^{N-1} \;+\; \int_{\{z<\zeta_* t\}\cap Q} \partial_t(s - s_*(\zeta_*))_+\, d\mathcal{L}^N \\ &\le \int_{\{z=\zeta_* t\}\cap Q} \Big\{\zeta_*\,(s - s_*(\zeta_*))_+ \;-\; (g(s) - g(s_*(\zeta_*)))_+\, u\cdot\nu_* \\ &\quad - (f(s) - f(s_*(\zeta_*)))_+\, \rho\Big\} d\mathcal{H}^{N-1}. \end{aligned}$$

Likewise, we have

$$\begin{aligned}
&\frac{d}{dt}\int_{\{z>\zeta_* t\}\cap Q}(s_*(\zeta_*)-s)_+\,d\mathcal{L}^N\\
&\leq \int_{\{z=\zeta_* t\}\cap Q}\Big\{-\zeta_*\,(s_*(\zeta_*)-s)_+ \;+\; (g(s_*(\zeta_*))-g(s))_+\,u\cdot\nu_*\\
&\quad +\; (f(s_*(\zeta_*))-f(s))_+\,\rho\Big\}d\mathcal{H}^{N-1}.
\end{aligned}$$

We add both

$$\begin{aligned}
&\frac{d}{dt}\Big[\int_{\{z<\zeta_* t\}\cap Q}(s-s_*(\zeta_*))_+\,d\mathcal{L}^N+\int_{\{z>\zeta_* t\}\cap Q}(s_*(\zeta_*)-s)_+\,d\mathcal{L}^N\Big]\\
&\leq -\int_{\{z=\zeta_* t\}\cap Q}\Big\{-\zeta_*(s-s_*(\zeta_*)) \;+\; (g(s)-g(s_*(\zeta_*)))\,u\cdot\nu_*\\
&\quad +\; (f(s)-f(s_*(\zeta_*)))\,\rho\Big\}d\mathcal{H}^{N-1}
\end{aligned}$$

and integrate over $\zeta_*\in\mathbb{R}$

$$\begin{aligned}
&\frac{d}{dt}\int_Q\left\{\int_\zeta^{+\infty}(s-s_*(\zeta_*))_+\,d\zeta_*+\int_{-\infty}^{\zeta}(s_*(\zeta_*)-s)_+\,d\zeta_*\right\}d\mathcal{L}^N\\
&\leq -\frac{1}{t}\int_Q\{-\zeta\,(s-s_*) \;+\; (g(s)-g(s_*))\,u\cdot\nu_* \;+\; (f(s)-f(s_*))\,\rho\}\,d\mathcal{L}^N,
\end{aligned}$$

where $\zeta=z/t$. The result now follows from the following three lemmas.

Lemma 19. *For $\zeta\in\mathbb{R}$ and $s\in[0,1]$ we have*

$$\begin{aligned}
&\int_\zeta^{+\infty}(s-s_*(\zeta_*))_+\,d\zeta_*+\int_{-\infty}^{\zeta}(s_*(\zeta_*)-s)_+\,d\zeta_*\\
&=\lambda\,m(s)\left|\frac{(\frac{1}{m})(s)-(\frac{1}{m})(s_*(\zeta))}{\frac{1}{m_h}-\frac{1}{m_l}}\right|^2+(\zeta-h'(1))_+\,(1-s)+(h'(0)-\zeta)_+\,s.
\end{aligned}$$

Lemma 20. *For $\zeta\in\mathbb{R}$, $s\in[0,1]$ and $u\in\mathbb{R}^N$ we have*

$$\begin{aligned}
-\zeta\,(s-s_*(\zeta)) \;+\; (g(s)-g(s_*(\zeta)))\,u\cdot\nu_* \;+\; (f(s)-f(s_*(\zeta)))\,\rho&\\
\geq\frac{1}{\lambda}\,\Big(m(s)\,|\tilde f|^2 \;+\; \tilde f\cdot\tilde u\Big)&,
\end{aligned}$$

where

$$\tilde u \;:=\; u-c\,\nu_* \quad \textit{and} \quad \tilde f \;:=\; \lambda\;\frac{(\frac{1}{m})(s)-(\frac{1}{m})(s_*(\zeta))}{\frac{1}{m_h}-\frac{1}{m_l}}\,\nu_*. \tag{71}$$

Lemma 21. *(59) and (60) imply*

$$\int_Q \left\{ m(s)|\tilde{f}|^2 + \tilde{f} \cdot \tilde{u} \right\} \;\geq\; 0, \tag{72}$$

where $\tilde{u}$ and $\tilde{f}$ are defined like in (71).

The only place where (62) enters is through the following identity, which is needed in Lemma 19 and 20. It quantifies the strict convexity of $\frac{1}{m}$.

Lemma 22. *For $s, s_* \in [0,1]$ we have*

$$(\tfrac{1}{m})(s) - (\tfrac{1}{m})(s_*) - (\tfrac{1}{m})'(s_*)\,(s - s_*) \;=\; m(s)\,\left((\tfrac{1}{m})(s) - (\tfrac{1}{m})(s_*)\right)^2. \tag{73}$$

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