Dimension Estimates for Invariant Sets of Dynamical Systems

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Abstract We investigate the relationships between various types of dimension like characteristics (e.g. Hausdorff, box, and Lyapunov dimensions, topological entropy) of dynamical systems and the geometric complexity of invariant and limit sets (e.g. isolated equilibria, global attracting or unstable cycles, products of Cantor-type sets and regular objects). We consider both flows and discrete-time dynamical systems of smoothness C^1 on n-dimensional smooth Riemannian manifolds which possess compact invariant sets. We formulate our results in terms of the singular values of the linearized evolution operator and consider additionally global informations such as the homology group and curvature properties of the manifold, natural Lyapunov functions and Losinskii norms, and the degree of non-injectivity of the generating map.

1 Introduction

The first general result for upper Hausdorff dimension estimates of flow invariant sets in $\mathbb{R}^n$ in terms of singular values of the linearization are given in [8]. This approach was extended in [47,31,4,51] to map-invariant sets on Riemannian manifolds and in [32,34] by including Lyapunov functions into the contraction conditions for outer Hausdorff measures. In [10,55] the results by Douady-Oesterlé were extended to estimates for invariant sets of evolution systems in Hilbert spaces. Hausdorff dimension estimates for invariant sets of general flows using the eigenvalues of the symmetric part of the linearization are deduced in [54] for the $\mathbb{R}^n$ and in [47,35,37] for manifolds. Douady-Oesterlé-type estimates for piecewise smooth maps on manifolds are given in [51,46] (see also [53]). The hyperbolic or quasi-hyperbolic structure of the invariant set was used in dimension estimates in [12,22,16] where also an entropy term was introduced into the dimension bound.
Several dimension upper bounds of invariant sets allow conclusions on the dynamical behaviour of the system. The key step in the papers [54,33,34,47] is to use properties of sets with Hausdorff dimension less than two. One can show ([54,5]) that such a compact invariant set contains no simple closed piecewise smooth invariant curves. On the basis of Hausdorff dimension estimates a generalization of certain global stability results for ODE's of Hartman-Olech and Borg ([25]), but also of other types of classical results from the Bendixson-Poincaré theory were derived in [34,41,42].

Parallel to Hausdorff dimension estimates a number of upper and lower bounds for the box dimensions of invariant sets were deduced in [4,60,29,2,36,43,55,26]. The box dimension of a set is never smaller than the Hausdorff dimension and gives important informations about the possibility to use embedding strategies. Recently it was shown ([14]) that certain embedding homeomorphisms can be chosen generically with Hölder-Lipschitz continuous inverse which enable conclusions for dimension estimates.
Hausdorff and box dimension estimates for flow invariant sets are very effective if appropriate types of local, global and uniform Lyapunov exponents are introduced ([10,31,32,55,60]). On the basis of such Lyapunov exponents various types of Lyapunov dimensions of a dynamical system with respect to an invariant set were defined (Kaplan-Yorke formula [18]) and it was conjectured that in typical cases these dimensions coincide with the Hausdorff dimension of the invariant set.
Parallel to the dimension and stability investigation of invariant sets of discrete-time and continuous-time dynamical systems various types of dimensions of invariant measures have been developed ([31,48,59]). Defining for the invariant ergodic measure of a dynamical system the Lyapunov exponents one can introduce the Lyapunov dimension of this measure which is an upper bound of the Hausdorff dimension of the measure. As in the measure free case various stability and convergence properties of the underlying dynamical system may be derived from the magnitude of the Lyapunov exponents of the measure. It is shown in [9] that if the invariant measure of a flow is ergodic and all Lyapunov exponents of this measure are negative, the support of the measure is a stable equilibrium point. If exactly one exponent of such a measure is zero and the remaining ones are negative, the support of the measure is a stable equilibrium or a stable limit cycle.
An important class of invariant sets of dynamical systems have locally the structure of a product of a smooth (often one-dimensional) submanifold directed “along the attractor” and a set, “transversal” to the attractor. Thus, it is naturally to investigate the dimension of such sets and stability of orbits of such attractors considering surfaces which are transversal to the flow-lines. The use of transversal intersections is well-known in stability investigations of flow lines: contracting or expanding behaviour in sections transversal to the flow line directions is the main reason for the stability or instability of the orbit ([25,34,37]).
The paper is organized as follows. In Section 2 we present a short review on attractors and Lyapunov functions. In Section 3 we introduce singular values of differentials, the covariantly written variational equation and basic facts on Losinskii measures and measure-free Lyapunov exponents. In Section 4 we give the definitions of various types of dimensions and of the topological entropy. A number of upper and lower dimension estimates (especially of Hausdorff dimension) of invariant sets are contained in Section 5. In Section 6 upper Hausdorff dimension bounds for flow-negatively invariant sets are formulated in terms of the eigenvalues of the symmetric part of an associated

system in normal variations. In Section 7 we give upper Hausdorff dimension estimates for systems with an equivariant tangential bundle splitting. As a special case flows on manifolds with negative curvature are considered. In the last Section 8 the cardinality of pre-image sets is used to get dimension estimates for non-injective maps.

2 Attractors, Dissipativity and Lyapunov Functions

Suppose M is an n-dimensional C^k-manifold with Riemannian metric g. Let on M be defined a dynamical system $\{\varphi^t\}_{t\in\Gamma}$ with $\Gamma \in \{\mathbb{Z}_+, \mathbb{Z}, \mathbb{R}, \mathbb{R}_+\}$. In the discrete-time case ($\Gamma \in \{\mathbb{Z}_+, \mathbb{Z}\}$) we assume that the dynamical system $\{\varphi^t\}_{t\in\Gamma}$ is generated by a sufficiently smooth map $\varphi \equiv \varphi^1 : M \to M$. In the continuous-time case ($\Gamma \in \{\mathbb{R}_+, \mathbb{R}\}$) we assume a smooth vector field $f : M \to TM$ as generator of $\{\varphi^t\}_{t\in\Gamma}$. If $\Gamma = \mathbb{R}$ we speak also about a *flow*. If $\Gamma \in \{\mathbb{R}, \mathbb{Z}\}$ we call the system *invertible*. Further we use the notion $\Gamma_+ := \Gamma \cap \mathbb{R}_+$.

A subset $A \subset M$ is called *invariant* with respect to $\{\varphi^t\}_{t\in\Gamma}$ if $\varphi^t(A) = A$ for all $t \in \Gamma$ and *positively invariant* if $\varphi^t(A) \subset A$ for all $t \in \Gamma_+$. A compact set $B_0 \subset M$ is *absorbing* for $\{\varphi^t\}_{t\in\Gamma}$ if the following properties hold:

1) $\varphi^t(B_0) \subset B_0$ for all $t \in \Gamma_+$.
2) For every bounded set $B \subset M$ there exists $t_B \in \Gamma_+$ such that $\varphi^t(B) \subset B_0$ for all $t \in \Gamma_+$ with $t \geq t_B$.

A system with a compact absorbing set is also called *dissipative*. A set $E \subset M$ is called *minimal* for $\{\varphi^t\}_{t\in\Gamma}$ if E is non-empty, closed, and invariant, and has no proper subsets with the same property.

Suppose $\{\varphi^t\}_{t\geq 0}$ is dissipative with respect to the bounded set B_0. Then B_0 contains a minimal set. To see this, note that since B_0 is bounded, the non-wandering set $\Omega(\varphi)$ of $\varphi = \{\varphi^t\}_{t\geq 0}$ is non-empty, closed, φ^t-invariant, and $\Omega(\varphi) \subset B_0$. Moreover, $\Omega(\varphi)$ is bounded. But any bounded, closed invariant set contains a minimal set.

A set $A \subset M$ is called *attractor* for $\{\varphi^t\}_{t\in\Gamma}$ if A is compact, invariant and if there exists an open set $U \supset A$ such that for any $\varepsilon > 0$ there is a time $t_\varepsilon \in \Gamma_+$ with $\varphi^t(U) \subset N_\varepsilon(A)$ for all $t \in \Gamma_+$ with $t \geq t_\varepsilon$. The set $W^s(A) = \{p \in M : \varphi^t(p) \to A \text{ for } t \to \infty\}$ is called *domain of attraction*. A set A is a *global attractor* for $\{\varphi^t\}_{t\in\Gamma}$ if it is an attractor and for any bounded set $B \subset M$ holds dist $(\varphi^t(B), A) \to 0$ for $t \to \infty$. If $\{\varphi^t\}_{t\in\Gamma}$ has a compact absorbing set B_0 then $A = \bigcap\limits_{t\in\Gamma_+} \varphi^t(B_0)$ is the global attractor for $\{\varphi^t\}_{t\in\Gamma}$.

An effective tool for the characterization of attractors or dissipativity regions are Lyapunov functions. For a given dynamical system $\varphi = \{\varphi^t\}_{t\in\Gamma}$ on M a smooth function $v : U \subset M \to \mathbb{R}$ is called *Lyapunov function* in $U_1 \subset U$ if $\mathcal{L}_\varphi v(p) \leq 0$ for all $p \in U_1$. Here $\mathcal{L}_\varphi v$ denotes the *Lie operator* of v with respect to $\{\varphi^t\}_{t\in\Gamma}$, defined as

$$\mathcal{L}_\varphi v(p) = \begin{cases} v(\varphi(p)) - v(p) & \text{if } p, \varphi(p) \in U_1 \text{ (discrete-time case)}, \\ \dot{v}(p) = \dfrac{d}{dt} v(\varphi^t p)_{|t=0} & \text{if } \varphi^t p \in U_1, |t| < \varepsilon \text{ (continuous-time case)}. \end{cases}$$

Lyapunov functions are important in the description of attractors and absorbing domains. It is well-known ([3]), that if A is an attractor for the flow $\{\varphi^t\}_{t\in\Gamma}$ on M, then there exists a C^k-Lyapunov function $v : W^s(A) \to \mathbb{R}_+$ such that $v(p) = 0$ on A, $v(p) > 0$ and $\dot{v}(p) < 0$ on $W^s(A)\backslash A$ and $v(p) \to +\infty$ for $p \to \partial W^s(A)$.
Hence, any attractor of a given flow may be described by means of a Lyapunov function v as $\{p \in M : v(p) = 0\}$, which shows ([3,21])that any attractor has the shape of a polyhedron.
We now describe a useful method for verifying the existence of Lyapunov functions for a dynamical system $\{\varphi^t\}_{t\in\Gamma}$ which shall be used in the following. In the continuous-time case we assume for this that the flow is given by a vector field in *feedback-control* form

$$\dot{x} = Ax + bu,\ u = \phi(c^* x)\ , \tag{1}$$

where $A : E_n \to E_n$, $b : E_m \to E_n$, and $c : E_l \to E_n$ are linear operators, $\phi : U \subset E_l \to E_m$ is a C^1-function (E_k denotes an Euclidean space of dimension k).
In the discrete-time case we consider the system

$$x \mapsto Ax + bu,\ u = \phi(c^* x)\ , \tag{2}$$

where A, b, c, and ϕ are as above. We say that the pair (A, b) of (1), respectively (2), is *stabilizable* if there exists an operator h such that the spectrum of $A + b\,h^*$ is in the continuous-time case contained in $\{z \in \mathbb{C} : \text{Re } z < 0\}$ and in the discrete-time case contained inside the complex unit circle.
The following theorem is proved by V.A.Yakubovich ([58]) and is called *frequency-domain lemma.*

Theorem 1. *Consider a dynamical system $\{\varphi^t\}_{t\in\Gamma}$ of (1) or (2) and assume that (A, b) is stabilizable. Let $F : E_n \times E_m \to \mathbb{R}$ be a quadratic form. Then there exists a Lyapunov function $v(x) = \langle x, Hx\rangle$ generated by an operator $H = H^* : E_n \to E_n$ and satisfying $\mathcal{L}_\varphi v(x, u) < 0$ for $x \in E_n, u \in E_m$ with $F(x, u) \geq 0$ and $\|x\| + \|u\| > 0$, if and only if there exists a $\delta > 0$ such that $F^c(x, u) \leq -\delta(\|x\|^2 + \|u\|^2)$ for all $x \in E_u^c$, $u \in E_m^c$ such that $\lambda x = Ax + bu$ for all $\lambda \in S$, where $S = \{i\omega : \omega \in \mathbb{R}\}$ in the continuous-time case and $S = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ in the discrete-time case. (Here is $\mathcal{L}_\varphi v(x, u) = 2x^* H(Ax + bu)$ in the continuous-time case and $\mathcal{L}_\varphi v(x, u) = (Ax + bu)^* H(Ax + bu) - x^* Hx$ in the discrete-time case. F^c denotes the Hermitean extension.)*

Remark 2. Using the frequency-domain lemma and a procedure which is called *averaging over the group* we can construct Lyapunov function for factor manifolds. Consider a dynamical system $\{\varphi^t\}_{t\in\Gamma}$ on M which is equivariant with respect to a finite group G of homeomorphisms of M, i.e. for any $t \in \Gamma$ and $g \in G$ we have $\varphi^t \circ g = g \circ \varphi^t$. Suppose also that $v : M \to \mathbb{R}$ is a Lyapunov function for φ on M, i.e. $\mathcal{L}_\varphi v(p) \leq 0$ on M. We want to consider the dynamical system on the factor-set M/G, which is assumed to have the structure of a factor manifold. We define a new scalar-valued function on M by $w(p) := \frac{1}{|G|}\sum_{g\in G} v(gp)$. It is easy to see that

$$w(gp) = \frac{1}{|G|}\sum_{g'\in G} v(g'gp) = w(p) \quad \text{for any } g \in G \text{ and}$$

$$w(\varphi^t p) - w(p) = \frac{1}{|G|}\sum_{g\in G} v(g\varphi^t p) - v(gp) = \frac{1}{|G|}\sum_{g\in G} v\,\underbrace{(\varphi^t(gp)) - v(g(p))}_{\leq 0}$$

for $p \in M$ and $t \in \Gamma$. Thus w has all properties of a Lyapunov function on M/G.

3 Multilinear Operators Acting on Bundles

3.1 Singular Values of Operators Acting Between Different Spaces

Let E and E' be two n-dimensional Euclidean spaces with the scalar products $\langle\cdot,\cdot\rangle_{\mathrm{E}}$ and $\langle\cdot,\cdot\rangle_{\mathrm{E}'}$, respectively. If $L : E \to E'$ is a linear map, we denote by $L^{[*]} : E' \to E$ the unique linear operator satisfying $\langle L^{[*]}u, v\rangle_{\mathrm{E}} = \langle u, Lv\rangle_{\mathrm{E}'}$, for all $u \in E'$ and $v \in E$. $L^{[*]}$ is called $\langle\cdot,\cdot\rangle_{\mathrm{E}},\ \langle\cdot,\cdot\rangle_{\mathrm{E}'}$-*adjoint*, or shortly *adjoint* ([20]). If $L = L^{[*]}$ the operator is called *self-adjoint*. Using the definition it is easy to see that $L^{[*]}L$ is self-adjoint.

Example 3. Consider the special case $L : (\mathbb{R}^n, \langle\cdot,\cdot\rangle_{\mathrm{G}_1}) \to (\mathbb{R}^n, \langle\cdot,\cdot\rangle_{\mathrm{G}_2})$. We identify L with its matrix in the canonical basis and assume scalar products given by positive-definite matrices G_1 and G_2. Then $L^{[*]} = G_1^{-1}L^*G_2$, where L^* denotes the usual transpose of the matrix L. It follows that $L^{[*]}L = G_1^{-1}L^*G_2L$ and $(L^{[*]}L)^{[*]} = L^{[*]}L$. The matrix $L^{[*]}L$ is non-negative, by which we understand that all eigenvalues of the generalized eigenvalue problem $L^*G_2L = \lambda G_1$ are non-negative.

The *singular values* of a linear operator $L : E \to E'$, denoted by $\sigma_1(L) \geq \ldots \geq \sigma_n(L)$ are defined to be the eigenvalues of the positive part of L, i.e. of the positive semi-definite operator $(L^{[*]}L)^{\frac{1}{2}} : E \to E$. (They are ordered with respect to their size and algebraic multiplicity). The definition of the numbers $\sigma_i(L)$ agrees with the min-max characterization (Courant-Hilbert theorem) of the eigenvalues of the operator $(L^{[*]}L)^{\frac{1}{2}}$: For every integer $m \in \{1, \dots, n\}$ we have

$$\sigma_m(L) = \sup_{\substack{F \subset E \\ \dim F = m}} \inf_{\substack{v \in F \\ \|v\|_{\mathrm{E}}=1}} \|Lv\|_{\mathrm{E}'} \quad . \tag{3}$$

Using the singular values of $L : E \to E'$ we can define the absolute value of the determinant by $|\det L| = [\det L^{[*]}L]^{\frac{1}{2}}$. The geometric meaning of the singular values is the following (see [55]). If $L : E \to E'$ has the singular values $\sigma_1(L) \geq \dots \geq \sigma_n(L) > 0$ and B is a ball in E of radius r then LB is an ellipsoid in E' with the length of semi-axes $\sigma_i r$, $i = 1, \dots, n$.
With $E^{\wedge k}$ or $\bigwedge^k E$ we denote the k-th exterior power of the n-dimensional Euclidean space E and we define the k-th *exterior power* of L by $L^{\wedge k} : E^{\wedge k} \to E^{'\wedge k}$ and the k-th *additive compound operator* by $L_k := \frac{d}{ds}(\mathrm{id} + sL)^{\wedge k}|_{s=0}$.
It is easy to see that $(L^{\wedge k})^{[*]} = (L^{[*]})^{\wedge k}$ for $k = 1, \dots, n$. If L is unitary, i.e. L is bijective and $L^{-1} = L^{[*]}$, then $L^{\wedge k}$ is also unitary.
By means of the singular values $\sigma_1(L) \geq \dots \geq \sigma_n(L)$ for an arbitrary integer $k \in \{0, 1, \dots, n\}$ we define

$$\omega_k(L) := \begin{cases} \sigma_1(L) \cdot \ldots \cdot \sigma_k(L) = \|L^{\wedge k}\|_{\mathrm{op}} & \text{for } k > 0 \ , \\ 1 & \text{for } k = 0 \ . \end{cases}$$

Further, for a number $d \in (0, n]$ written as $d = d_0 + s$ ($d_0 \in \{0, 1, \dots, n-1\}$, $s \in (0, 1]$) we put

$$\omega_d(L) := \omega_{d_0}^{1-s}(L) \cdot \omega_{d_0+1}^{s}(L) = \|L^{\wedge d_0}\|_{\mathrm{op}}^{1-s} \|L^{\wedge (d_0+1)}\|_{\mathrm{op}}^{s} \quad .$$

We indicate that the generalized *Horn's inequality* holds ([46,47]):
Suppose that E, E' and E'' are n-dimensional Euclidean spaces, $L : E \to E'$ and $L' : E' \to E''$ are linear operators, and $d \in [0, n]$ is an arbitrary number. Then $\omega_d(L'L) \leq \omega_d(L') \cdot \omega_d(L)$.

3.2 The Vertical Part of the Variational Equation and Liouville-Inequalities

For a given smooth dynamical system $\{\varphi^t\}_{t \in \Gamma}$ on the smooth n-dimensional Riemannian manifold (M, g) let us consider the linearization, i.e. the first order Taylor expansion

$$\varphi^t(\exp_p v) = \exp_{\varphi^t p}((d_p \varphi^t)v + o(v)) \tag{4}$$

with $p \in M$, $v \in T_pM$, $t \in \Gamma$, $|t| \leq c$. ($\exp_p$ denotes the exponential map around p.) If the system is time-continuous the differential of the time t-map is described by the covariantly written variational equation with respect to the infinitesimal generator f of $\{\varphi^t\}_{t \in \mathbb{R}}$,

$$f(x) := \frac{d}{dt}\varphi^t(x)|_{t=0} \ \in T_xM \quad .$$

The variational equation with respect to the motion $t \mapsto \varphi^t p$ is given by

$$\frac{Dy}{dt} = \nabla f(\varphi^t p) y \ , \tag{5}$$

where the linear operator $\nabla f(p) : T_pM \to T_pM$ is the *covariant derivative* of f in p, defined in local coordinates with the help of the Christoffel symbols Γ_{ij}^k by $v = v^i \partial_i(p) \in T_pM \mapsto \nabla_i f^k v^i \partial_k(p)$ with

$$\nabla_i f^k = \frac{\partial f^k}{\partial x^i} + \Gamma_{ij}^k f^j \ . \tag{6}$$

$\frac{Dy}{dt}$ denotes the absolute derivative along the integral curve $t \mapsto \varphi^t(p)$ and is in local coordinates of a chart x given by $\frac{Dy^i}{dt} = \nabla_l f^i y^l$. In order to investigate the stability of orbits and the deformation of k-forms in the tangent space we consider the operator $S\nabla f(p) := \frac{1}{2}[\nabla f(p) + \nabla f(p)^{[*]}] : T_pM \to T_pM$, which in a chart x is given by the matrix $S_m^i = \frac{1}{2}[g^{ij}\nabla_j f^k g_{km} + \nabla_m f^i]$. The eigenvalues of the operator $S\nabla f(p)$ are defined as the eigenvalues of the symmetric quadratic form *(deformation tensor)*

$$e_{si} = \tfrac{1}{2}[f_{s,i} + f_{i,s}] \ \text{ with } f_{s,i} = g_{st}\nabla_i f^t \ , \tag{7}$$

i.e. λ is a zero of the equation $\det[e_{si} - \lambda g_{si}] = 0$. In general, for an arbitrary number $k \in \{1, \ldots, n\}$, we consider the *k-th compound variational equation*

$$\frac{Dw}{dt} = [\nabla f(\varphi^t p)]_k w \ , \tag{8}$$

where $[\nabla f(u)]_k : \bigwedge^k T_uM \to \bigwedge^k T_uM$ is the k-th additive compound of $\nabla f(u)$.

If $y_1(\cdot), \ldots, y_k(\cdot)$ are solutions of the variational equation (5) with respect to $\varphi^t(p)$ then $y_1(\cdot) \wedge \ldots \wedge y_k(\cdot)$ is a solution of the k-th compound variational equation (8) since

$$\frac{D}{dt}[y_1(t) \wedge \ldots \wedge y_k(t)] = \frac{Dy_1(t)}{dt} \wedge \ldots \wedge y_k(t) + \ldots + \ y_1(t) \wedge \ldots \wedge \frac{Dy_k(t)}{dt} \ .$$

Suppose that in the time-continuous case $\sigma_i(d_u\varphi^t)$, $i = 1, 2, \ldots, n$, are the singular values of $d_u\varphi^t$ for $u \in M$ and $t \in \mathbb{R}_+$. Assume further that $\alpha_1(u) \geq \ldots \geq \alpha_n(u)$ are the eigenvalues of the operator $S\nabla f(u)$. Then for any $t \geq 0$ the generalized Liouville inequalities have the form ([47])

$$\underbrace{\sigma_1(d_u\varphi^t) \cdot \ldots \cdot \sigma_k(d_u\varphi^t)}_{\omega_k(d_u\varphi^t)} \leq \exp \int_0^t [\alpha_1(\varphi^\tau u) + \ldots + \ \alpha_k(\varphi^\tau u)] d\tau \tag{9}$$

and

$$\sigma_n(d_u\varphi^t)\cdot\ldots\cdot\sigma_{n-k+1}(d_u\varphi^t) \geq \exp\int_0^t [\alpha_n(\varphi^\tau u)+\ldots+\alpha_{n-k+1}(\varphi^\tau u)]d\tau \quad . \tag{10}$$

As a special case we get the equality $|\det(d_u\varphi^t)| = \exp\int_0^t \operatorname{div} f(\varphi^\tau u)d\tau$.

3.3 Losinskii Measures on Bundles

In order to improve the contraction properties of the singular value function with respect to a given metric we also consider on M, parallel to the Riemannian structure, a Banach space structure in the tangent spaces T_uM, given by a family $\{\|\cdot\|_u\}$ of norms on TM.
Let us consider at first the general situation $T: E \to E$, where T is a linear operator on the n-dimensional space E with norm $\|\cdot\|_{\mathrm{E}}$. The functional $m: L(E,E) \to \mathbb{R}$ defined by

$$m(T) := \lim_{s\to 0+0} \frac{\|id_E + sT\|_{\mathrm{op}} - 1}{s}$$

is called ([6]) *Losinskii measure* (*Losinskii norm* or *logarithmic norm*). The Losinskii measure has several basic properties which follow directly from its definition:

$$\begin{array}{ll} (1) & m\,(S+T) \leq\ m(S)+m(T) \quad \text{for} \quad S,T \in L(E,E) \ . \\ (2) & m\,(\alpha T) =\ \alpha m(T) \quad \text{for} \quad \alpha \geq 0 \ . \\ (3) & m\,(T) \leq\ \|T\|_{\mathrm{op}} \ . \end{array}$$

Instead of the generalized Liouville inequalities, using the supplementary Finsler structure on the Riemannian manifold (i.e. the family of norms $\{\|\cdot\|_u\}$ on the tangent bundle TM), we get the following inequalities: Suppose that $m(\cdot)$ is a Losinskii measure on the tangent bundle. Then there exists (see [5]) a continuous function $\rho: M \to \mathbb{R}^+$ such that for any $k \in \{1,\ldots,n\}$, $t \geq 0$, and $u \in M$ the inequalities

$$\sigma_1(d_u\varphi^t)\cdot\ldots\cdot\sigma_k(d_u\varphi^t)\frac{\rho(\varphi^t u)}{\rho(u)} \leq \exp\int_0^t m[\nabla f(\varphi^\tau u)]_k d\tau \tag{11}$$

and

$$\frac{\rho(\varphi^t u)}{\rho(u)}\sigma_n(d_u\varphi^t)\cdot\ldots\cdot\sigma_{n-k+1}(d_u\varphi^t) \geq \exp\int_0^t -m[-\nabla f(\varphi^\tau u)]_k d\tau \tag{12}$$

are satisfied.
Note that in the special case, where the norms are generated by the Riemannian structure, it holds $m([\nabla f(\varphi^t u)]_k) = \alpha_1(d_u\varphi^t)+\ldots+\alpha_k(d_u\varphi^t)$, and choosing $\rho(u)\equiv 1$, we get the inequalities (9) resp. (10).

3.4 Lyapunov Exponents

Suppose that $\{\varphi^t\}_{t\in\Gamma}$ is a smooth injective dynamical system on the Riemannian manifold M. Let $\sigma_1(d_u\varphi^t) \geq \ldots \geq \sigma_n(d_u\varphi^t) > 0$ denote the singular values of $d_u\varphi^t : T_uM \to T_{\varphi^t u}M$, i.e. the eigenvalues of the positive definite operator $([d_u\varphi^t]^{[*]}[d_u\varphi^t])^{\frac{1}{2}} : T_uM \to T_uM$. Let $K \subset M$ be a compact invariant set of $\{\varphi^t\}_{t\in\Gamma}$. Define for any $j \in \{1,\ldots,n\}$ and $t \in \Gamma_+$

$$\omega_{j,K}(\varphi^t) = \sup_{u\in K} \omega_j(d_u\varphi^t) \text{ and } \quad \omega_{0,K}(\varphi^t) = 1 \ . \tag{13}$$

For any $j \in \{1,\ldots,n\}$ and $u \in M$ the *local Lyapunov exponents* $\nu_j(u)$ of $\{\varphi^t\}_{t\in\Gamma}$ are defined iteratively by [10,55,56]

$$\nu_1(u) + \ldots + \nu_j(u) = \limsup_{t\to\infty} \frac{1}{t} \ \ln \omega_j(d_u\varphi^t) \ . \tag{14}$$

The *global Lyapunov exponents* $\nu_1,\ldots,\nu_n$ are defined by

$$\nu_1 = \lim_{t\to\infty} \frac{1}{t} \ln \omega_{1,K}(\varphi^t) \text{ and } \nu_j = \lim_{t\to\infty} \frac{1}{t} \ln \frac{\omega_{j,K}(\varphi^t)}{\omega_{j-1,K}(\varphi^t)} \tag{15}$$

for $j = 2,3,\ldots,n$. Note that these limits exist because of the subexponential property ([55]) of the singular value function. The *upper Lyapunov exponents* are given by

$$\overline{\nu}_k = \limsup_{t\to\infty} \frac{1}{t} \ln \sup_{u\in K} \ln \sigma_k(d_u\varphi^t) \quad (k = 1,2,\ldots,n) \ .$$

4 Basic Definitions from Dimension Theory

4.1 Topological, Hausdorff- and Box Dimensions

Let X be a topological space. We say that X is *finite dimensional* if there exists an integer $n \geq -1$ such that, for every open covering α of X, there exists another open covering β refining α such that every point of X belongs to at most $n+1$ sets of β. (Recall that a covering $\alpha = \{A\}$ of X is *open* if all sets A of α are open. A covering β of X *refines* another covering α of X if for any $B \in \beta$ there exists an $A \in \alpha$ such that $B \subset A$.) In this case the *topological dimension* of $X, \dim_{\mathrm{T}} X$, is defined as the minimal number n satisfying this property (see [30]). Then $\dim_{\mathrm{T}} \emptyset = -1, \dim_{\mathrm{T}} \mathbb{R}^n = n$, and if K is a compact finite dimensional space, it is homomorphic to a subset of $\mathbb{R}^n$ with $n = 2\dim_{\mathrm{T}} K + 1$.

Suppose that (X,ϱ) is a metric space, E is an arbitrary subset of X, $d \geq 0$ and $\varepsilon > 0$ are real numbers. Let us cover E by at most countable many balls of radii $r_j < \varepsilon$ and define

$$\mu_{\mathrm{H}}(E,d,\varepsilon) = \inf \sum_j r_j^d \ ,$$

where the infimum is taken over all such countable ε-covers of E under the convention that $\inf \emptyset = +\infty$. It is obvious that $\mu_{\mathrm{H}}(E,d,\varepsilon)$ increases with decreasing ε. Thus, there exists the limit (which may be infinite)

$$\mu_{\mathrm{H}}(E,d) = \lim_{\varepsilon \to 0+0} \mu_{\mathrm{H}}(E,d,\varepsilon) \ ,$$

which is called the outer *Hausdorff-d-measure* of E. For any fixed $E \subset X$ there exists a critical value $d_{\mathrm{cr}} \in [0,+\infty]$ such that

$$\mu_{\mathrm{H}}(E,d) = \begin{cases} +\infty & \text{for all } 0 \le d < d_{\mathrm{cr}}, \\ 0 & \text{for all } d > d_{\mathrm{cr}} \ . \end{cases}$$

This critical value is defined as the *Hausdorff dimension* $\dim_{\mathrm{H}} E$ of E.
Let (X,ϱ) be a metric space, $K \subset X$ a compact subset. For any $\varepsilon > 0$ denote by $N_\varepsilon(K)$ the minimal number of balls of radius ε which are necessary to cover K. Define the *box dimension* (or *upper box, fractal, entropic dimension*) of K, and the *lower box dimension* of K by

$$\overline{\dim}_{\mathrm{B}} K = \limsup_{\varepsilon \to 0+0} \frac{\ln N_\varepsilon(K)}{-\ln \varepsilon} \quad \text{and} \quad \underline{\dim}_{\mathrm{B}} K = \liminf_{\varepsilon \to 0+0} \frac{\ln N_\varepsilon(K)}{-\ln \varepsilon} \ ,$$

respectively.
It is well-known ([48]) that for any compact set K it holds

$$\dim_{\mathrm{T}} K \le \dim_{\mathrm{H}} K \le \underline{\dim}_{\mathrm{B}} K \le \overline{\dim}_{\mathrm{B}} K \ .$$

4.2 Topological Pressure and Topological Entropy

Suppose that M is an n-dimensional Riemannian manifold and $\{\varphi^t\}_{t\in\Gamma}$ is a dynamical system on M. For $g \in C(M)$, $p \in M$ and $t \ge 1$ define $S_t g(p) = \sum_{i=0}^{t-1} g(\varphi^i p)$ in the discrete-time case, and $S_t g(p) = \int_0^t g(\varphi^s p) ds$ in the continuous-time case.
In the discrete-time case a set $E \subset M$ is (ε, T)-*separated* with respect to $\{\varphi^k\}_{k\in\Gamma}$ if $p,q \in E, p \ne q$, imply that $\varrho(\varphi^k p, \varphi^k q) > \varepsilon$ for some integer $k \in \{0,\ldots,T-1\}$. In the continuous-time case a set $E \subset M$ is (ε,T)-*separated* with respect to $\{\varphi^t\}_{t\in\Gamma}$ if $p,q \in E, p \ne q$, imply $\varrho(\varphi^t p, \varphi^t q) > \varepsilon$ for some $t \in [0,T]$. For a dynamical system $\varphi = \{\varphi^t\}_{t\in\Gamma}$ on M, $g \in C(M)$, $\varepsilon > 0$ and $T \in \Gamma, T > 0$, define

$$Z_{\mathrm{T}}(\varphi,g,\varepsilon) = \sup_{E:(\varepsilon,T)-\text{separated}} \left\{ \sum_{p\in E} \exp S_T g(p) \right\}$$

and $$P(\varphi,g,\varepsilon) = \limsup_{T\to\infty} \frac{1}{T} \ln Z_{\mathrm{T}}(\varphi,g,\varepsilon) \ .$$

Then there exists the limit $P(\varphi,g) = \lim_{\varepsilon\to 0} P(\varphi,g,\varepsilon)$ which is called ([48,57]) *topological pressure* of g.

In particular, taking $g \equiv 0$, we get the *topological entropy* $h_{\mathrm{top}}(\varphi) = P(\varphi, 0)$. For expanding and hyperbolic systems which are both expansive ([52,57]) we have the following property. There is some $\varepsilon_0 > 0$ such that for any $\varepsilon < \varepsilon_0$ and any continuous function g

$$P(\varphi, g) = \limsup_{T\to\infty} \frac{1}{T} \ln Z_{\mathrm{T}}(\varphi, g, \varepsilon) \ .$$

5 Dimension Estimates of Invariant Sets

5.1 Upper Hausdorff and Box Dimension Estimates

Suppose there is given a C^1-map $\varphi : U \subset M \to M$ on an open set U of an n-dimensional Riemannian C^k-manifold. Further we assume that $K \subset U$ is a φ-invariant compact set. In order to estimate various types of dimension of this set we consider the singular values $\sigma_1(d_u\varphi) \geq \ldots \geq \sigma_n(d_u\varphi) \geq 0$ of the differential $d_u\varphi : T_uM \to T_{\varphi(u)}M$ in any point $u \in U$ and for a fixed $d = d_0 + s$ ($d_0 \geq 0$ integer, $s \in (0,1]$) the singular-value function

$$\omega_{d,K}(\varphi) = \sup_{u\in K} \omega_d(d_u\varphi), \text{ where } \omega_d(d_u\varphi) = \omega_{d_0}^{1-s}(d_u\varphi)\omega_{d_0+1}^{s}(d_u\varphi) \ .$$

For $p \in U$ the *local Lyapunov dimension of* φ at p denoted by $\dim_{\mathrm{L}}(\varphi, p)$ is defined to be the largest number $d \in [0, n]$ such that $\omega_d(d_p\varphi) \geq 1$. If $\sigma_1(d_p\varphi) < 1$ let $\dim_{\mathrm{L}}(\varphi, p) = 0$. For a compact set $K \subset U$ let $\dim_{\mathrm{L}}(\varphi, K) = \max \dim_{\mathrm{L}}(\varphi, p)$ be the *Lyapunov dimension* of φ on K ([18,29]).
A basic result for various upper dimension bounds is the following theorem ([47,46,31,4,51]). Note that the introduction of a Lyapunov function in the dimension estimate for the phase space $M = \mathbb{R}^n$ was first done in [33].

Theorem 4. *Suppose that $\varphi : U \subset M \to M$ is as above, $K \subset U$ is a φ-invariant compact set, $d \in (0, n]$ is a fixed number and $\rho : U \to \mathbb{R}_+$ is a continuous function such that*

$$\sup_{u\in K} \frac{\rho(\varphi(u))}{\rho(u)} \omega_d(d_u\varphi) < 1 \ . \tag{16}$$

Then $\dim_{\mathrm{H}} K \leq d$.

Remark 5. The basic constructions of the proof of Theorem 4 are the following. There exists an $\varepsilon > 0$ such that the exponential map $\exp_p$ is an embedding of the closed ε-ball $\overline{B_\varepsilon(0_p)} = \{v \in T_pM : \|v\| \leq \varepsilon\}$ into M for every point $p \in K$. This gives closed neighbourhoods $V_p = \overline{B_\varepsilon(p)}$ of $p \in K$. The local coordinates describing V_p depend continuously on p. If $\varrho(p, q) < \delta$ we can use the radial isometry of $\exp_p$, i.e. we can pass to coordinates on V_p in which $q \in V_p \subset M$ corresponds to a point $z \in T_pM$ with $\|z\|_{T_pM} = \varrho(p, q)$.

Using these properties of the exponential map we can fix a $D \in [0,n]$ and show that for any $b_0 > 2\sqrt{n}$ and $C_0 > 2^D n^{D/2}$ there is an $\varepsilon_0 > 0$ such that for all $u \in K$ and all $A \subset B_{\varepsilon_0}(u)$

$$\mu_{\rm H}(A, D, b\varepsilon) \le C\mu_{\rm H}(\varphi(A), D, \varepsilon)$$

for all $\varepsilon < \varepsilon_0$, where $C = C(C_0, n, D)$ and $b = b(C_0, n, D)$.

Remark 6. It is useful to compare condition (16) with the following situation. If $\varphi : M \to M$ is a diffeomorphism on the manifold M and $\rho : M \to \mathbb{R}_+$ is the density of a φ-invariant measure with respect to the canonical volume form then

$$\frac{\rho(\varphi(u))}{\rho(u)}\omega_n(d_u\varphi) \equiv 1 \ .$$

To illustrate how the frequency-domain lemma (Theorem 1) can facilitate the application of Theorem 4 we consider the special case of a map in the feedback-control form (2). The following result was shown by A. Noack in [45].

Corollary 7. *Consider the map (2) and assume that $K \subset D \subset E_n$ (D open) is a compact invariant set for (2). Assume further that the pairs (A, b) and (A^*, c) from (2) are stabilizable and that for certain real numbers $d \in (0, n]$, $\kappa > 0$ and $\lambda > 0$ the following conditions hold:*

(1) *All eigenvalues of $\frac{1}{\kappa}A$ lie outside the complex unit circle.*

(2) $\frac{1}{2}\eta^*([\phi'(c^*x)]^* + \phi'(c^*x))\eta \le \lambda\|\eta\|^2$ *for all* $\eta \in E_m$ *and* $x \in D$.

(3) $Re\ \chi(\kappa z) + \lambda\chi^*(\kappa z)\chi(\kappa z) \le 0$ *for all* $z \in \mathbb{C}$ *with* $|z| = 1$.

(4) $\frac{1}{\kappa^{n-d}}\det[A + b\phi'(c^*x)c^*] < 1$ *for all* $x \in D$.

Then $\dim_{\rm H} K < d$.

We now formulate a direct consequence of Theorem 4 for the Hausdorff dimension estimation of flow-invariant sets (see [47,5]). Note that using the results of [19] the following bound is also true for the box dimension.

Corollary 8. *Suppose that the flow $\{\varphi^t\}_{t\in\mathbb{R}}$ on the n-dimensional manifold M is generated by the C^1-vector field f and $\alpha_1(u) \ge \ldots \ge \alpha_n(u)$ are the eigenvalues of the symmetrized covariant operator $S\nabla f(u) : T_uM \to T_uM$. Suppose that there is an integer $d_0 \in (0, n-1]$ and a real number $s \in (0,1]$ such that for a $t > 0$*

$$\sup_{u\in K}\int_0^t [\alpha_1(d_u\varphi^\tau) + \ldots + \alpha_{d_0}(d_u\varphi^\tau) + s\ \alpha_{d_0+1}(d_u\varphi^\tau)]d\tau < 0 \ . \qquad (17)$$

If K is a compact set and $\varphi^t(K) \supset K$, then $\dim_{\rm H} K \le d_0 + s$.

Proof. For small t the singular values of the operator $d_u\varphi^t : T_uM \to T_{\varphi^t u}M$ are the eigenvalues of the matrix

$$\sqrt{[\mathrm{id}_{T_uM} + tB(x^k) + o\,(t,x^k)]^{[*]}\,[\mathrm{id}_{T_uM} + tB\,(x^k) + o\,(t,x^k)]},$$

where $B(x^k) = \nabla_i f^i(x^k)$ is the representation of the covariant derivative in a chart x. Since

$$\begin{aligned}\left[\mathrm{id}_{T_uM} + tB(x^k) + o\,(t,x^k)\right]^{[*]}\,\left[\mathrm{id}_{T_uM} + tB(x^k) + o\,(t,x^k)\right] = \\ = (\mathrm{id}_{T_uM} + \tfrac{1}{2}t\,[B^{[*]}(x^k) + B(x^k)])^2 + o\,(t,x^k)\end{aligned}$$

we get for $i = 1, 2, \ldots, n$

$$\sigma_i(d_u\varphi^t) = \sqrt{(1 + t\alpha_i(u))^2 + o\,(t,u)} = 1 + t\alpha_i(u) + o\,(t,u)$$

and, consequently,

$$\begin{aligned}\omega_d(d_u\varphi^t) = (1 + t\alpha_1(u)) \cdot \ldots \cdot (1 + t\alpha_{d_0}(u))(1 + t\alpha_{d_0+1}(u))^s + o\,(t,u) = \\ 1 + t[\alpha_1(u) + \ldots + \alpha_{d_0}(u) + s\,\alpha_{d_0+1}(u)] + o\,(t,u)\ .\end{aligned}$$

□

Remark 9. Let us consider the special case of a smooth vector field f on the manifold $M = \mathbb{R}^n$ with the metric $(g_{ij}(x)) = \rho^2(x)I$, where $\rho : \mathbb{R}^n \to \mathbb{R}_+$ is a C^1-function. To verify condition (17) we have to compute the eigenvalues of the deformation tensor e_{ij}, defined in (7), which is in this case $\frac{1}{2}\,[J+J^*]+\frac{\rho'}{\rho}I$, where $\frac{1}{2}\,[J + J^*]$ is the symmetrized Jacobi matrix. If we take $\rho(x) = e^{\frac{v(x)}{d_0+s}}$, where v is a C^1-function, we have $\frac{\rho'}{\rho} = \frac{\dot{v}}{d_0+s}$. It follows that if $\alpha_i(x)$ are the eigenvalues of $\frac{1}{2}\,[J(x) + J(x)^*] + \frac{\dot{v}(x)}{d_0+s}$ and $\widetilde{\alpha}_i(x)$ are the eigenvalues of $\frac{1}{2}\,[J(x) + J(x)^*]$, then $\alpha_i(x) = \widetilde{\alpha}_i(x) + \frac{\dot{v}(x)}{d_0+s}$ for $i = 1, 2, \ldots, n$.

For embedding techniques it is very important to have not only upper bounds for the Hausdorff dimension but also for the box dimension. The next theorem gives such an upper bound ([4]) under the additional condition (1), which, as was shown later in [29,2,19], is not necessary .

Theorem 10. *Suppose that $\varphi : U \subset M \to M$ is defined as in Theorem 4, $K \subset U$ is a compact φ-invariant set, and $\rho : U \to \mathbb{R}_+$ is a continuous function. Suppose further that the following conditions are satisfied:*

(1) $\omega_n(d_u\varphi) = \mathrm{const} \neq 0$ *for all* $u \in K$.
(2) *There exists a number* $s \in (0,1]$ *such that* $\sup\limits_{u\in K} \frac{\rho(\varphi(u))}{\rho(u)}\omega_{n-1+s}(d_u\varphi) < 1$.

Then $\overline{\dim}_{\mathrm{B}}K \leq n - 1 + s$. *Further if* $\rho(u) \equiv 1$ *then* $\overline{\dim}_{\mathrm{B}}K \leq \dim_{\mathrm{L}}(\varphi, K)$.

The full proof of Theorem 10 is given in [4]. Let us only note here that for the proof of the theorem the new metric $\widetilde{g}_{ij}(x) = \rho^2(x) g_{ij}(x)$ on M is considered. Note that the transition to an equivalent metric in M does not alter the box dimension. Thus, the singular-value function which is taken with respect to a fixed Riemannian metric, can be replaced by the singular-value function with respect to an adapted metric $\widetilde{g}$.

Example 11. Consider a mapping which describes a model for repeated impacts of a bouncing ball on a sinusoidally vibrating table and which is given by the map $\varphi_{\alpha,\gamma} : \mathbb{R}^2 \to \mathbb{R}^2$ through

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x+y \\ \alpha y - \gamma \cos(y+x) \end{pmatrix} . \tag{18}$$

(See also [7].) Here γ plays the role of force amplitude for the table motion and $0 < \alpha \le 1$ is the dissipation. An approach of the dynamics can be found in [23].

For any pair of parameters (α, γ) the map $\varphi_{\alpha,\gamma}$ is a diffeomorphism. The determinant of its Jacobian matrix is equal to α, which means that $\varphi_{\alpha,\gamma}$ generates a dissipative dynamical system for $\alpha < 1$. For that case all orbits enter and remain within the strip bounded by $y = \pm\gamma/(1-\alpha)$. Since (18) is invariant under coordinate change $x \mapsto x + 2\pi n$ for $n = \pm 1, \pm 2, \dots$ the phase space can be interpret to be the cylinder $S^1 \times \mathbb{R}$. Furthermore, there is an compact attracting set K within the absorbing set ($\varepsilon > 0$ small) $D \subset \{(x,y) : |y| \le \varepsilon + \gamma/(1-\alpha)\} \subset S^1 \times \mathbb{R}$. With the standard metric for factor manifolds given by $(g_{ij}(x)) = I$ in any chart, $S^1 \times \mathbb{R}$ is a 2-dimensional smooth Riemannian manifold, the flat cylinder. The largest singular value of $d_p\varphi_{\alpha,\gamma}$ for $p = (x,y) \in S^1 \times \mathbb{R}$ with respect to this metric is given by

$$\sigma_1(d_p\varphi_{\alpha,\gamma}) = \frac{1}{2}\left(\sqrt{(\alpha+1)^2 + r(x,y)} + \sqrt{(\alpha-1)^2 + r(x,y)}\right) ,$$

where $r(x,y) = 2\gamma^2 \sin^2(x+y) + 2\gamma \sin(x+y)\alpha + 1$, and $\sigma_1(d_p\varphi_{\alpha,\gamma}) > 1$ for all $p \in S^1 \times \mathbb{R}$ and all $\alpha \in (0,1)$. Estimating the box dimension Theorem 10 yields

$$\overline{\dim}_{\mathrm{B}} K \le 1 + \left(1 - \frac{\ln\alpha}{\ln\left(\sqrt{(1+\alpha)^2 + b} + \sqrt{(1-\alpha)^2 + b}\right) - \ln 2}\right)^{-1} ,$$

with $b = 2\gamma^2 - 2\gamma + 1$. For parameter values $\alpha = 0.5, \gamma = 10$ considered in [23] we obtain $\overline{\dim}_{\mathrm{B}} K \le 1.790$.

Example 12. We consider the Hénon map $\varphi_{\alpha,\beta} : \mathbb{R}^2 \to \mathbb{R}^2$ given by ([27])

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \alpha + \beta y - x^2 \\ x \end{pmatrix} \tag{19}$$

with parameters $0 < \alpha, 0 < \beta < 1$. The map has the two fixed points $p_\pm = (x_\pm, y_\pm)$ with $y_\pm = x_\pm = \frac{1}{2}(\beta - 1 \pm \sqrt{(\beta-1)^2 + 4\alpha})$. Define a function $v : \mathbb{R}^2 \to \mathbb{R}$ by $v(x,y) = \gamma(x + \beta y)$ for $\gamma = \left[(-2x_- + \beta - 1)\left(\sqrt{x_-^2 + \beta}\right)\right]^{-1}$ and introduce the metric tensor g in the standard chart around $p = (x,y)$ as $e^{2(1-s)v(x,y)}\begin{pmatrix}\sqrt{\beta} & 0 \\ 0 & 1\end{pmatrix}$ for some $s \in (0,1)$. The singular values with respect to this metric are given for $p = (x,y)$ by

$$\sigma_{1/2}(d_p\varphi_{\alpha,\beta}) = e^{(1-s)\gamma(\alpha+(\beta-1)x-x^2)}\left(\sqrt{x^2+\beta} \pm |x|\right) \ .$$

Consider the case $\beta = 0.3$, $\alpha = 1.4$. For any $\varphi_{\alpha,\beta}$-invariant compact set $K \subset \mathbb{R}^2$ Theorem 10 gives $\overline{\dim}_{\mathrm{B}} K \le 1.496$. The result agrees with the one obtained in [36], where the estimate is obtained using a coordinate transform and the function v as a Lyapunov-type function instead of making use of a certain metric. For every compact invariant set K which is located in the square $-1.8 \le x, y \le 1.8$ using the flat metric on $\mathbb{R}^2$ gives the estimate $\overline{\dim}_{\mathrm{B}} K \le 1.523$ (see [29]). We refer to [45] for another application on choosing Riemannian metrics which implies even smaller bounds. Following the observation in [36] it can be seen that the global Lyapunov dimension $\dim_{\mathrm{L}}(\varphi_{\alpha,\beta}, K)$ is equal to the local Lyapunov dimension $\dim_{\mathrm{L}}(\varphi_{\alpha,\beta}, p_-)$.

Example 13. We consider the *complex Lorenz equation* which is introduced in [15] and which may be written as

$$\begin{aligned} \dot{x} &= -c\,(x - y), \\ \dot{y} &= -xz + rx - ay, \\ \dot{z} &= -bz + \tfrac{1}{2}\,(x^* y + x y^*) \ . \end{aligned} \tag{20}$$

In this system c and b are positive parameters and $a = 1 - ie$ resp. $r = r_1 + ir_2$ are complex parameters ($e, r_1, r_2 \in \mathbb{R}$).
For $r_2 = e = 0$ we have the *real Lorenz equation.* If either Im $(r - a) \ne 0$ or Im $(r-a) = 0$ and Re $(r-a) \le 0$ system (20) has the only equilibrium point $(0,0,0)$. For Im $(r-a) = 0$ and Re$(r-a) > 0$ the system has a continuum of equilibrium points ([39,38]).
Writing $x = x_1 + ix_2$ and $y = y_1 + iy_2$ we obtain from (20) the real ODE system in $\mathbb{R}^5$

$$\begin{aligned} \dot{x}_1 &= -cx_1 + cy_1, \\ \dot{x}_2 &= -cx_2 + cy_2, \\ \dot{y}_1 &= -x_1 z + r_1 x_1 - r_2 x_2 - y_1 - e y_2, \\ \dot{y}_2 &= -x_2 z + r_2 x_1 + r_1 x_2 + e y_1 - y_2, \\ \dot{z} &= -bz + x_1 y_1 + x_2 y_2 \ . \end{aligned} \tag{21}$$

Note that if f denotes the right-hand side of (21) then div$f \equiv -2c - 2 - b < 0$. In order to prove the dissipativity of (21) we use a Lyapunov function as in [39].

Put $\kappa_0 = \min\{1, b, c\}$ and take real auxiliary constants $\kappa_1 \in (0, \kappa_0)$, $\kappa_2 > 0$ and κ_3 satisfying $\kappa_2\,(c - \kappa_1)(1 - \kappa_1) - \frac{1}{4}\,|c - \kappa_2 r - \kappa_3|^2 \geq 0$. Define also constants κ_4 and κ_5 by

$$\kappa_4 > \frac{\kappa_3^2 (b - 2\kappa_1)^2}{8\kappa_1\kappa_2(b - \kappa_1)} \quad \text{and} \quad \kappa_5 > \frac{\kappa_3^2}{\kappa_2(1 + \kappa_2)} \left(\frac{(b - 2\kappa_1)^2}{4\kappa_1(b - \kappa_1)} + 1 \right) \ .$$

Introduce now the function $v : \mathbb{R}^5 \to \mathbb{R}$ by

$$v\,(x_1, x_2, y_1, y_2, z) = \frac{1}{2}[x_1^2 + x_2^2 + \kappa_2(y_1^2 + y_2^2 + z^2)] - \kappa_3 z \ .$$

A straightforward calculation shows (see [39,38]) that for any bounded set $B \subset \mathbb{R}^5$ there exists a time $t = t_B$ such that for $t \geq t_B$

$$v(x_1(t), x_2(t), y_1(t), y_2(t), z(t)) < \kappa_4 \quad \text{and} \quad x_1^2(t) + x_2^2(t) < \kappa_5$$

holds provided $(x_1(0), x_2(0), y_1(0), y_2(0), z(0)) \in B$ and $t \geq t_B$.

It follows that the global attractor A of the Lorenz system (21) is contained in the ellipsoid

$$\mathcal{E} = \Big\{ (x_1, x_2, y_1, y_2, z) \in \mathbb{R}^5 : x_1^2 + x_2^2 + \kappa_2(y_1^2 + y_2^2) + \\ + \kappa_2 (z - \frac{\kappa_3}{\kappa_2})^2 < 2\kappa_4 + \frac{\kappa_3^2}{\kappa_2} \Big\} \ .$$

¿From this we get the estimates

$$y_l^2 \leq \frac{1}{\kappa_2} \left(2\kappa_4 + \frac{\kappa_3^2}{\kappa_2} \right), \quad l = 1, 2, \quad \text{and} \quad \left(z - \frac{\kappa_3}{\kappa_2} \right)^2 \leq \frac{1}{\kappa_2} \left(2\kappa_4 + \frac{\kappa_3^2}{\kappa_2} \right) \ .$$

Using the global Lyapunov exponents $\nu_1, \ldots, \nu_5$ with respect to A and the estimate of A given by $\mathcal{E}$ we obtain for the box dimension on the basis of Theorem 10 (see also [26])

$$\overline{\dim}_{\mathrm{B}} A \leq 4 + \frac{\nu_1 + \nu_2 + \nu_3 + \nu_4}{|\nu_5|} \leq 4 + \frac{C}{-\mathrm{div} f + C} \ ,$$

where $\nu_1 + \nu_2 + \nu_3 + \nu_4 \leq C$ and $|\nu_5| = -\nu_5 = \nu_1 + \nu_2 + \nu_3 + \nu_4 - \mathrm{div} f$.

The bound C can be estimated by means of the variational equation, using the ellipsoid $\mathcal{E}$. In [24] for parameter values $(r_1, r_2, e, c, b) = (60, 0.02, 0.06, 2, 0.8)$ the estimate $\dim_{\mathrm{H}} A \leq 4.46$ for the global attractor A of (21) was given. On the basis of Theorem 10 this constant bounds also the box dimension of the attractor A.

5.2 Dimension Estimates and the Spanning Surface Principle

In this subsection we use upper dimension estimates of invariant sets of dynamical systems for the characterization of the underlying dynamics. Using such an approach it is possible to get higher-dimensional generalizations of certain theorems of Bendixson-Poincaré type and global convergence results. The key step is to exclude the existence of closed orbits of the flow by using a spanning surface argument. In the ODE case for $M = \mathbb{R}^n$ such an approach is developed in several papers ([41,33,34]). In [47] the analog problem is considered for vector fields on manifolds. In the 1-dimensional version of invariant objects the "spanning surface argument" means that for any closed invariant curve we consider spanning surfaces and argue that 2-dimensional measures of such surfaces are bounded from below. We suppose that the n-dimensional smooth manifold M has this property, i.e. if there is a smooth closed curve then there exists a two-dimensional surface in M with finite 2-dimensional Lebesque-measure such that it spans the given curve. We can use in this situation the generalization of a result from [41] which can be formulated as follows. Suppose that $D = \{(u,v) \in \mathbb{R}^2 : u^2 + v^2 < 1\}$ is the open disc. On Lip $(D \to M)$, i.e. on the space of Lipschitz continous functions from D in M, we consider the area functionals

$$A(h) = \mathrm{mes}_2\, h\,(D) \ .$$

If x is a chart in $h\,(D)$, $x(u,v)$ is the coordinate representation in this chart of $x \circ h$ and $g_{ij}(x)$ is the local realization of the metric tensor, then this area functional is given by

$$\int\!\!\int_D \sqrt{EG - F^2}\, dudv \ ,$$

where $E = g_{ij}(x)x_u^i x_u^j$, $G = g_{ij}(x)x_v^i x_v^j$ and $F = g_{ij}(x)x_u^i x_v^j$. If c is a simple closed rectifiable curve in M and $\Sigma(c,M)$ is the set of all Lipschitz continuous maps h from $\overline{D} \subset \mathbb{R}^2$ in M satisfying $h(\partial D) = c(\partial D)$ it can be shown that there exists a $\delta > 0$ such that $A(h) \geq \delta$ for all $h \in \Sigma(c,M)$. (For the $\mathbb{R}^n$ case this is done in [41], the manifold case is investigated in [5]).
The next theorem can be considered as generalized negative Bendixson theorem for manifolds ([47]). Other versions of such generalizations are given in [5,33,34].

Theorem 14. *Suppose that on the simply-connected compact n-dimensional Riemannian C^m-manifold M ($m \geq 4$) there is defined a flow $\{\varphi^t\}_{t\in\mathbb{R}}$, generated by a smooth vector field f. Suppose also that $\alpha_1(u) \geq \alpha_2(u)$ are the two largest eigenvalues of the symmetrized covariant derivative $S\nabla f(u)$ such that $\alpha_1(u) + \alpha_2(u) < 0$ on M. Then $\{\varphi^t\}_{t\in\mathbb{R}}$ has on M no closed invariant curve.*

Proof. Suppose the opposite, i.e. suppose that there exists a closed invariant curve γ of $\{\varphi^t\}_{t\in\mathbb{R}}$. By assumption we can construct a surface given by a parametrization h, which spans γ, i.e. $h(\partial D) = \gamma$. We can now use the fact, as mentioned above, that there is a $\delta > 0$ such that $A(h) \geq \delta > 0$ for any such γ-spanning surface. Since $\varphi^t(\gamma) = \gamma$ for all $t \in \mathbb{R}$, we conclude by the smoothness of the flow that $h^t := \varphi^t \circ h$ is for any t also an integrable parametrization of a γ-spanning surface. This gives a contradiction, since by $\alpha_1(u) + \alpha_2(u) < 0$ on M it follows that $\mathrm{mes}_2(\varphi^t \overline{D}) \to 0$ as $t \to +\infty$. □

Remark 15. An analogous result is possible if we consider the existence of k-dimensional $(2 \leq k \leq n)$ invariant submanifolds for a given flow $\{\varphi^t\}_{t \in \mathbb{R}}$. If we require that for the largest eigenvalue $\alpha_1^{(k)}(u)$ of the symmetrized k-th compound operator $[S\nabla f]_k(u)$ the inequality $\alpha_1^{(k)}(u) < 0$ on M is satisfied and any that k-dimensional invariant submanifold may be written as boundary of a $(k+1)$-dimensional surface, then our dynamical system $\{\varphi^t\}_{t \in \mathbb{R}}$ can not have any such k-dimensional invariant submanifold on M. A basic idea for the proof is the use of Stokes theorem for the integration of k-forms ω^k on $(k+1)$-dimensional submanifolds U^{k+1} on M by $\int_{U^{k+1}} d\omega^k = \int_{\partial U^{k+1}} \omega^k$.

In the following we want to derive conditions for the global convergence of motions of a given flow to equilibrium points on the basis of Theorem 14. There we apply the C^1-closing lemma of Pugh ([49]), which we formulate for completeness in the following version.

Lemma 16. *Suppose that $\varphi = \{\varphi^t\}_{t \in \Gamma}$ is a C^1-dynamical system on an n-dimensional Riemannian C^m-manifold M. Let p_0 be a non-wandering point of φ such that the iterates $\{\varphi^t(p_0)_{t \in \Gamma}\}$ of p_0 are contained in a compact subset of M. Then $\{\varphi^t\}_{t \in \Gamma}$ can be C^1-approximated by a dynamical system $\{\psi^t\}_{t \in \Gamma}$ which has a periodic orbit through p_0.*

Now we can state a convergence theorem for flows on manifolds. (See also [46,47].)

Theorem 17. *Suppose that on a simply-connected compact n-dimensional Riemannian C^m-manifold M a smooth flow $\{\varphi^t\}_{t \in \mathbb{R}}$ is given, generated by the vector field f such that the conditions of Theorem 14 are satisfied. Suppose further that $\{\varphi^t\}_{t \in \mathbb{R}}$ has equilibrium points and these points are all isolated. Then any motion of $\{\varphi^t\}_{t \in \mathbb{R}}$ tends for $t \to +\infty$ to an equilibrium.*

Proof. Suppose that $t \mapsto \varphi^t(u)$ is an arbitrary motion of the flow. Since M is compact the ω-limit set $\omega(u) \neq \emptyset$. Suppose that there exists a point $p \in \omega(u)$ which is not an equilibrium point. Since such a point is non-wandering, due to Pugh's lemma there exists in any C^1-neighborhood of f a smooth vector field $\widetilde{f}$ such that the associated flow $\{\widetilde{\varphi}^t\}_{t \in \mathbb{R}}$ has a periodic motion through p. Since M is compact, and the conditions of Theorem 14 are satisfied, we can choose $\widetilde{f}$ such that the two largest eigenvalues $\widetilde{\alpha}_1(u) \geq \widetilde{\alpha}_2(u)$ of $S\nabla\widetilde{f}(u)$ satisfy the inequality $\widetilde{\alpha}_1(u) + \widetilde{\alpha}_2(u) < 0$ on M. On the basis of Theorem 14 we conclude that the vector field $\widetilde{f}$ cannot generate closed motions. It follows that p is an equilibrium point. Since $\omega(u)$ is connected and the equilibrium points are by assumption isolated, it follows that $\omega\,(u) = \omega\,(p)$, which shows that the motion through u converges. □

5.3 Lower Estimates for the Lower Box Dimension by Use of Lyapunov Functions

In a general not necessarily hyperbolic situation lower estimates of dimensions are very rare in the literature. Only for the lower box dimension sufficiently general estimates are known. A classical result goes back to G. Kushnirenko ([12]).

Theorem 18. *Suppose that M is an n-dimensional C^m-manifold ($m \geq 3$). Suppose further that $\varphi : M \to M$ is a C^1-map, $K \subset M$ is a compact φ-invariant set, and* $\sup_{u\in K} \|d_u\varphi\| > 1$. *Then*

$$\underline{\dim}_{\mathrm{B}} K \geq \frac{h_{\mathrm{top}}(\varphi_{|K})}{\ln \sup_{u\in K} \|d_u\varphi\|} \quad .$$

In the following two corollaries we give some technical realizations of such a lower bound, which are due in parts to A. Noack ([46]).

Corollary 19. *Assume that on the manifold M a smooth flow $\varphi = \{\varphi^t\}_{t\in\mathbb{R}}$ is given, which is generated by the vector field f and has a compact invariant set K. Suppose further that $v : D \to \mathbb{R}$ ($D \subset M$ open with $K \subset D$) is a C^1-function and $\alpha_1(u)$ is the largest eigenvalue of the symmetrized covariant derivative of f in u such that*
$\sup_{u\in K} [\alpha_1(u) + \dot{v}(u)] > 0$. *Then* $\underline{\dim}_{\mathrm{B}} K \geq \dfrac{h_{\mathrm{top}}(\varphi_{|K})}{\sup_{u\in K}[\alpha_1(u) + \dot{v}(u)]}$ *holds.*

In the next corollary we consider again a general dynamical system, but we assume that this system is injective.

Corollary 20. *Suppose that $\varphi = \{\varphi^t\}_{t\in\Gamma}$ is an injective dynamical system on M, having a compact invariant set K such that the first upper Lyapunov exponent $\overline{\nu}_1$ on K is positive. Then*

$$\underline{\dim}_{\mathrm{B}} K \geq \frac{h_{\mathrm{top}}(\varphi^1_{|K})}{\overline{\nu}_1} \quad .$$

Proof. For any positive integer k we have

$$h_{\mathrm{top}}({\varphi^1}_{|K}) = \frac{1}{k} h_{\mathrm{top}}({\varphi^k}_{|K}) \leq \frac{1}{k} \ln \sup_{u\in K} \sigma_1(d_u\varphi^k)\, \underline{\dim}_{\mathrm{B}} K \quad .$$

This implies

$$h_{\mathrm{top}}({\varphi^1}_{|K}) \leq \limsup_{k\to\infty} \frac{1}{k} \ln \sup_{u\in K} \sigma_1(d_u\varphi^k)\, \underline{\dim}_{\mathrm{B}} K \quad .$$

□

Remark 21. If μ is a Borel-measure, the *L-box dimension* of μ (named after F. Ledrappier ([31])), is

$$\dim_{\mathrm{L}}(\mu) = \lim_{\varepsilon \to 0} \limsup_{\delta \to 0} \frac{\ln N(\delta;\varepsilon)}{\ln 1/\varepsilon} \ ,$$

where $N(\delta;\varepsilon)$ is the minimal number of δ-balls needed to cover a compact set up to a μ-measure ε. It is easy to see that there is an analogy to the box dimension of a set ([48]): Suppose φ is C^1 on the compact Riemannian manifold M and μ is an invariant ergodic probability measure with $\lambda_1(\mu) > 0$ and metric entropy h_μ. Then $\dim_{\mathrm{L}}(\mu) \geq \frac{h_\mu}{\lambda_1(\mu)}$.

5.4 Topological Dimension Estimates of Minimal Sets of Flows

In difference to the great number of Hausdorff and box dimensions estimates analog results for the topological dimension of invariant sets are rare (of course, we always have the estimates $\dim_{\mathrm{T}} K \leq \dim_{\mathrm{H}} K \leq \underline{\dim}_{\mathrm{B}} K$). One of the few results in the literature concerning only the topological dimension goes back to G. F. Hilmy ([28]; see also [5]). We formulate and prove an adapted version for Riemannian manifolds.

Theorem 22. *Suppose that $\{\varphi^t\}_{t\in\Gamma}$ is a smooth flow on an n-dimensional Riemannian C^m-manifold M and $\Sigma \subsetneqq M$ is a minimal set of $\{\varphi^t\}_{t\in\Gamma}$. Then* $\dim_{\mathrm{T}} \Sigma \leq n-1$.

In order to prove this theorem we need two auxiliary results.

Lemma 23. *If $E \subset M$ is an invariant set for $\{\varphi^t\}_{t\in\Gamma}$ then the boundary ∂E is also invariant.*

Proof. Suppose $p \in \partial E$ is an arbitrary point, $t \in \mathbb{R}$ and $\varepsilon > 0$ are arbitrary numbers. Using the continuity of the map $\varphi^t : M \to M$ we can find a $\delta > 0$ such that $\varphi^t(B_\delta(p)) \subset B_\epsilon(\varphi^t(p))$. Since $B_\delta(p)$ contains points of E as well as points of $M \backslash E$, it follows that $\varphi^t(B_\delta(p))$ and, consequently, $B_\varepsilon(\varphi^t(p))$ has the same property. But $\varepsilon > 0$ is arbitrary, therefore $\varphi^t(p) \in \partial E$. □

Lemma 24. *If Σ is a minimal set of $\{\varphi^t\}_{t\in\Gamma}$ all its points are either boundary or inner points.*

Proof. Suppose that Σ has boundary points as well as inner points. Since Σ is closed, $\partial\Sigma \subset \Sigma$. By Lemma 23 the boundary $\partial\Sigma$ is a proper closed and invariant subset of Σ. But this contradicts the minimality of Σ. □

Proof (of Theorem 22). Suppose that $\dim_{\mathrm{T}} \Sigma = n$. Then ([30]) Σ contains an inner point. Thus, according to Lemma 24, all points of Σ are inner points. But this implies that Σ is open. Thus, $\Sigma \notin \{M, \emptyset\}$ is open and closed. But this is not possible, which shows that $\dim_{\mathrm{T}} \Sigma < n$. □

Remark 25. It is interesting to compare the estimate of the topological dimension given for Σ on the base of Theorem 22 with Theorem VIII 3' from [30]. In the latter theorem it is proved that in order that $\dim_{\mathrm{T}} C \leq n-1$ for a compact space X of a finite dimension it is necessary and sufficient that for the given closed subset C of X only the zero element of the homology group $H_{n-1}(C)$ bounds in X. (This means that the natural homomorphism of $H_{n-1}(C)$ in $H_{n-1}(X)$ must be an isomophism of $H_{n-1}(C)$ in $H_{n-1}(X)$.)

6 Upper Dimension Estimates for Semi-Flows in Normal Coordinates

6.1 The Variational System in Normal Coordinates

The aim of this subsection is to introduce a type of reparametrization of the neighborhood of flow lines, which goes back to [35] and which is very useful in stability and dimension theory.
Consider a semi-flow $\{\varphi^t\}_{t\geq 0}$ on an n-dimensional Riemannian C^m-manifold M generated by a vector field f of class C^2. For every regular point $u \in M$ with $f(u) \neq O_u$ (the origin of T_uM), we introduce the linear subspace of T_uM

$$T^{\perp}(u) := \{w \in T_uM : \langle w, f(u)\rangle = 0\} \ .$$

To describe how the orthogonal deviation of a perturbation in the initial conditions evolves in time we split a solution $y(\cdot)$ of (5) into orthogonal components $y(t) = z(t) + \mu(t) f(\varphi^t p)$, with $z(t) \in T^{\perp}(\varphi^t p)$ and $\mu(t)$ a time-dependent factor. Then $z(\cdot)$ is a solution of the *system in normal* or *transversal* variations

$$\frac{Dz}{dt} = A(\varphi^t p) z \tag{22}$$

where the linear operator $A(u) : T_uM \to T_uM$ is defined by

$$A(u)v = \nabla f(u)v - \frac{2\langle f(u), S\nabla f(u)v\rangle}{\|f(u)\|^2} f(u) \tag{23}$$

for all $v \in T_uM$. In local coordinates of a chart x around u the operator is given by the tensor

$$A_i^k = \nabla_i f^k - \frac{2}{g_{rs} f^r f^s} f^k g_{jl} f^l S_i^j \ \ ,$$

where f^k and g_{jl} are the coordinates of the vector field f and the metric tensor in the chart x, respectively, and $S_i^j = \frac{1}{2}(g^{ik}\nabla_k f^l g_{li} + \nabla_i f^j)$ is the representation of $S\nabla f(u)$ in this chart.
To formulate the following central result on the reparametrization of orbits in a general not necessarily hyperbolic situation, we need some notations. Recall

that since M is $C^m (m > 3)$ for any $p \in M$ there exists an open set $\Omega_p \subset T_pM$ around O_p such that the exponential map $\exp_p : \Omega_p \to \exp_p(\Omega_p) \subset M$ is a C^{m-2}-diffeomorphism. For $\delta > 0$ sufficiently small such that $B_\delta(O_p) \subset \Omega_p$ we consider the $(n-1)$-dimensional submanifold $B_\delta^\perp(p) := \exp_p(B_\delta(O_p)) \cap T^\perp(p)$ of M through p, where $B_\delta(O_p) \subset T_pM$ denotes a ball of radius δ centered in the origin of T_pM. Since every point $u \in B_\delta^\perp(p)$ is uniquely determined by the geodesic connecting u and p the point u can be described by the coordinates (r, v) such that $u = \exp_p(r, v)$, where $v \in T^\perp(p)$ is a vector of length 1 and $r \in [0, \delta)$ measures the arc length along the geodesic.

To investigate the stability of a fixed orbit or the dimension properties of a flow invariant set it suffices to consider stability or deformation properties of the projection of the flow lines transversally to the considered orbits or a fixed orbit of the invariant set, resp..

Keeping this in mind several methods of reparametrization of flow lines are of great importance ([25,34]). The following result goes back to [37]. (See also [46].)

Lemma 26. *Suppose $\varphi^{(\cdot)}(p)$ with $p \in M$ to be a non-constant solution of f with bounded positive semi-orbit $\gamma_+(p)$ such that for a certain $C_0 > 0$ the inequality $\|f(u)\| > C_0$ is satisfied for all $u \in \gamma_+(p)$. Then it holds: For any finite time $T_0 > 0$ there exists a $\delta = \delta(T_0) > 0$ such that for any pair $(r, v) \in [0, \delta] \times (T^\perp(p) \cap \partial B_1(O_p))$ we may find a C^{m-1}-diffeomorphism $s(\cdot, r, v) : \mathbb{R}_+ \to \mathbb{R}_+$ with $s(t, r, v) = t$ for all $t \in \mathbb{R}_+$ such that near $\gamma_+(p)$ the reparametrized flow $\phi(t, r, v) = \varphi^{s(t,r,v)}(\exp_p(rv))$ of f satisfies the condition*

$$\langle D_2\phi(t, r, v), f(\phi(t, r, v))\rangle = 0$$

for all $t \in [v, T_0]$.

6.2 Upper Hausdorff Dimension Estimates in Normal Coordinates

In this subsection we give Hausdorff dimension bounds of flow invariant sets of $\{\varphi^t\}_{t \geq 0}$, generated by f, which are formulated in terms of the eigenvalues of the symmetric part of the operator which generates the associated system in normal variations with respect to the direction of the vector field.

In the following we denote at any regular point p of f the eigenvalues of the operator $SA(p)$ restricted to the subspace $T^\perp(p)$ by $\beta_1(p) \geq \ldots \geq \beta_{n-1}(p)$. By $Z(t, p)$ we denote the operator solution of (22) satisfying $Z(0, p) = \mathrm{id}_{T^\perp(p)}$. For every $t \in \mathbb{R}_+$ the linear operator $Z(t, p) : T^\perp(p) \to T^\perp(\varphi^t p)$ maps between the subspaces $T^\perp(p)$ and $T^\perp(\varphi^t p)$ being orthogonal to the vector field in p and $\varphi^t p$, respectively.

The next lemma which can be proved analogously to the inequality (9) (see [37]) shows the distortion of the d-dimensional ellipsoid-measure under the linearized flow of f.

Lemma 27. *Suppose that $p \in M$ is a regular point of f and $Z(\cdot, p)$ is the operator solution of the equation in transversal variations (22). If $d \in (0, n-1]$ is an arbitrary number, written as $d = d_0 + s$ with integer d_0 and $s \in (0, 1]$ for $t \geq 0$ it holds*

$$\omega_d(Z(t,p)) \leq \exp\left\{ \int_0^t [\beta_1(\varphi^\tau p) + \ldots + \beta_{d_0}(\varphi^\tau p) + s\beta_{d_0+1}(\varphi^\tau p)] d\tau \right\} \ .$$

Remark 28. For a fixed regular point $p \in M$, small $r > 0$ and $t \geq 0$ the set $\mathcal{E}(t) := Z(t,p)B_r^\perp(O_p)$ is an ellipsoid in the subspace $T^\perp(\varphi^t p)$. If the numbers $\sigma_1(\mathcal{E}(t)) \geq \ldots \geq \sigma_{n-1}(\mathcal{E}(t))$ are the lengths of the semi-axes of $\mathcal{E}(t)$ and $d \in (0, n-1]$ is an arbitrary number we have $\omega_d(\mathcal{E}(t)) = \omega_d(Z(t,p))r^d$.

The next theorem which goes back to K. Gelfert (see [35]) is the realization of the Douady-Oesterlé estimate of Theorem 4 adapted to our special system of normal coordinates.

Theorem 29. *Let K and $\widetilde{K}$ two compact sets in M satisfying the inclusions $K \subset \varphi^t(K) \subset \widetilde{K}$ for all $t \geq 0$, where K does not contain equilibrium points of $\{\varphi^t\}$. Denote by Λ the set of equilibrium points of $\{\varphi^t\}$ in M. For $p \in \widetilde{K} \backslash \Lambda$ let $\beta_1(p) \geq \ldots \geq \beta_{n-1}(p)$ be the eigenvalues of the symmetric part $SA(p)$ restricted to the subspace $T^\perp(p)$, where $A(p)$ is the operator from (23). Let $v : M \backslash \Lambda \to \mathbb{R}$ be a C^1-function and suppose that there exist real numbers $d \in (0, n-1]$ $(d = d_0 + s,\ s \in (0,1],\ d_0$ integer), $\Theta > 0$ and $T > 0$ such that*

$$\sup_{p \in \widetilde{K} \backslash \Lambda} \int_0^T [\beta_1(\varphi^\tau p) + \ldots + \beta_{d_0}(\varphi^\tau p) + s\beta_{d_0+1}(\varphi^\tau p) + \dot{v}(\varphi^\tau p)] d\tau \leq -\Theta \ .$$

Then $\dim_{\mathrm{H}} K \leq d + 1$. *Further if $d = 1$ we have* $\dim_{\mathrm{H}} K \leq 1$.

6.3 Transversal Lyapunov Exponents for Semi-Flows

We now introduce Lyapunov exponents with respect to the variational equation in normal coordinates. Suppose again that $K \subset M$ is a compact invariant set of the semi-flow $\{\varphi^t\}_{t \geq 0}$ and in K there are no equilibrium points of this flow. For points in K we introduce the tangent bundle splitting $T_K M = E^1 \oplus E^2$ with $E_p^1 = T^\perp(p)$ and $E_p^2 = T''(p) = \mathrm{span}\{f(p)\}$ and require that the equivariance property $E^i_{\varphi^t p} = d_p\varphi^t E_p^i$ $(i = 1, 2,\ p \in K,\ t \geq 0)$ is satisfied. Denote the *transversal singular* values of the differential $d_p\varphi^t{}_{|E_p^1}$ by $\sigma_1^\perp(d_p\varphi^t) \geq \ldots \geq \sigma_{n-1}^\perp(d_p\varphi^t)$ and the *transversal singular value function* with respect to $\sigma_i^\perp(d_p\varphi^t)$ by $\omega_k^\perp(d_p\varphi^t)$. Analogously to the definitions in Section 3.4 let us introduce global transversal Lyapunov exponents by the following procedure. Define for $t \geq 0$ and $j = 1, 2, \ldots, n-1$ the numbers

$$\omega_{j,K}^{\perp}(\varphi^t) = \sup_{p \in K} \omega_j^{\perp}(d_p\varphi^t) \text{ and } \omega_{0,K}^{\perp}(t)(\varphi^t) \equiv 1 \ .$$

Then the *global transversal Lyapunov exponents* are

$$\nu_j^{\perp} = \lim_{t \to \infty} \frac{1}{t} \ln \frac{\omega_{j,K}^{\perp}(\varphi^t)}{\omega_{j-1,K}^{\perp}(\varphi^t)}, \quad j = 1, \ldots, n-1 \ . \tag{24}$$

With respect to Theorem 29 we have now the following

Corollary 30. *Suppose under the above assumptions that M is an n-dimensional Riemannian manifold $(n \geq 2), \{\varphi^t\}_{t \geq 0}$ is a semi-flow on M with invariant compact set K, and the global transversal Lyapunov exponents are defined by (24). Suppose further that there is a number $d_0 \in \{1, \ldots, n-2\}$ such that $\nu_1^{\perp} + \ldots + \nu_{d_0}^{\perp} < 0$ and $\nu_1^{\perp} + \ldots + \nu_{d_0+1}^{\perp} > 0$.*
Then $\dim_{\mathrm{H}} K \leq d_0 + 1 + \dfrac{\nu_1^{\perp} + \ldots + \nu_{d_0}^{\perp}}{|\nu_{d_0+1}^{\perp}|}$.

7 Upper Hausdorff Dimension Estimates for Dynamical Systems with an Equivariant Tangent Bundle Splitting

We introduce now a structure which is naturally given for hyperbolic systems. But in contrast to these systems we do not require any norm contraction or expansion properties with respect to a bundle of invariant subspaces. On an n-dimensional Riemannian manifold we consider an invertible dynamical system $\{\varphi^t\}_{t \in \Gamma}$ having a compact invariant set K. We say that $\{\varphi^t\}_{t \in \Gamma}$ possesses an *equivariant tangent bundle splitting* $T_K M = E^+ \oplus E^- \oplus E^0$ if for any $p \in K$ the set $E_p^+ := E^+ \cap T_p M$ is an n^+-dimensional subspace of $T_p M$, the set $E_p^- := E^- \cap T_p M$ is an n^--dimensional subspace of $T_p M$ and $E_p^0 := E^0 \cap T_p M$ is an n^0-dimensional subspace of $T_p M$ such that $n^- + n^+ + n^0 = n$ and $d_p\varphi^t(E_p^{\xi}) = E_{\varphi^t p}^{\xi}$ for $t \in \Gamma$ and $\xi \in \{+, -, 0\}$.

Example 31. Hyperbolic sets of flows allow a specific type of equivariant splitting of the tangent bundle, if we consider the stable part E^s as E^+, the unstable part E^u as E^- and the central part E^c as E^0.

We want to consider systems with a certain contraction property of the singular values of the tangent map. In order to use the splitting structure of the tangent bundle we define a singular function in a modified form. Suppose that there is an equivariant tangent bundle splitting as above, the numbers $\sigma_1^+(d_p\varphi^t), \ldots, \sigma_{n^+}^+(d_p\varphi^t)$ are the singular values of $d_p\varphi^t{}_{|E_p^+}$ and the numbers

$\sigma_1^-(d_p\varphi^{-t}), \ldots, \sigma_{n^-}^-(d_p\varphi^{-t})$ are the singular values of $d_p\varphi^{-t}_{|E_p^-}$. For arbitrary $p, q \in K$ and $t > 0$ relabel the set of singular values

$$\{\sigma_1^+(d_p\varphi^t), \ldots, \sigma_{n^+}^+(d_p\varphi^t), \sigma_1^-(d_q\varphi^{-t}), \ldots, \sigma_{n^-}^-(d_q\varphi^{-t})\}$$

as $\{\sigma_1(t,p,q), \sigma_2(t,p,q), \ldots, \sigma_{n^++n^-}(t,p,q)\}$ with $\sigma_1(t,p,q) \geq \ldots \geq \sigma_{n^++n^-}(t,p,q)$ and define the *singular value function* with respect to the splitting $(E^+,\, E^-,\, E^0)$ and $d = d_0 + s \in (0, n^+ + n^-]$ by

$$\omega_{d,K}^{E^+,\, E^-,\, E^0}(\varphi^t) = \begin{cases} 1 \\ \sup_{p,q\in K} \sigma_1(t,p,q) \cdot \ldots \cdot \sigma_{d_0}(t,p,q)\sigma_{d_0+1}^s(t,p,q) \end{cases}$$

for $d = 0$ resp. $d > 0$.

We say that φ^t is *d-contractive* with respect to the splitting $E^+ \oplus E^- \oplus E^0$ and $t \in \Gamma$ if it holds $\omega_{d,K}^{E^+,\, E^-,\, E^0}(\varphi^t) < 1$.

Since $\omega_{d,K}^{E^+,\, E^-,\, E^0}(\varphi^t)$ is subexponential in t, the limit

$$\overline{\nu}_d = \lim_{t\to\infty} \frac{1}{t} \ln \omega_{d,K}^{E^+,\, E^-,\, E^0}(\varphi^t)$$

exists for all $d \in [0, n^- + n^+]$. The numbers $\nu_1, \ldots, \nu_{n^++n^-}$, computed by the formula $\nu_i = \overline{\nu}_i - \overline{\nu}_{i-1}$ $(i = 1, 2, \ldots, n^+ + n^-)$ with $\overline{\nu}_0 = 0$, are the *global Lyapunov exponents of the splitting* (E^+, E^-, E^0). The following theorem goes back to A. Franz ([16]) and generalizes results of [12,22].

Theorem 32. *Let M be an n-dimensional Riemannian C^m-manifold, $\{\varphi^t\}_{t\in\Gamma}$ a smooth invertible dynamical system, and $K \subset M$ a compact φ^t-invariant set. Suppose that there exist an equivariant splitting $T_K M = E^+ \oplus E^- \oplus E^0$ of dimensions (n^+, n^-, n^0) and a number $d \in (0, n^+ + n^-]$ such that φ^t is d-contractive with respect to the splitting. Let us assume that there exists a time $t \in \Gamma$ such that*

$$\omega_{d,K}^{E^+,\, E^-,\, E^0}(\varphi^t) < e^{-2h_{\mathrm{top}}(\varphi^t{}_{|K})} \ .$$

Then $\dim_{\mathrm{H}} K \leq d + n_0$.

Corollary 33. *Let M be an n-dimensional Riemannian C^m-manifold, $U \subset M$ be an open set, $\{\varphi^t\}_{t\in\mathbb{R}}$ a C^1-flow on U and $K \subset U$ a compact φ^t-invariant set which admits an equivariant splitting $T_K M = E^+ \oplus E^- \oplus E^0$ of dimensions (n^+, n^-, n^0). Let $\nu_1, \ldots, \nu_{n^++n^-}$ be the global Lyapunov exponents of $\{\varphi^t\}_{t\in\mathbb{R}}$ on K with respect to this splitting and let $d_0 < n^+ + n^-$ be the smallest natural number with*

$$2h_{\mathrm{top}}(\varphi^1{}_{|K}) + \nu_1 + \ldots + \nu_{d_0} + \nu_{d_0+1} < 0 \ .$$

Then $\dim_{\mathrm{H}} K \leq d_0 + \dfrac{2h_{\mathrm{top}}(\varphi^1{}_{|K}) + \nu_1 + \ldots + \nu_{d_0}}{|\nu_{d_0+1}|} + n_0.$

Example 34. We consider the geodesic flow on a smooth n-dimensional Riemannian manifold, i.e. the flow defined on the unit tangent bundle $T^1M = \{v \in TM : \|v\| = 1\}$ such that the projection π of any flow line $t \mapsto \varphi^t v$ with $v \in TM$ onto M is the geodesic $\gamma_v(t)$ which is determined by $\gamma_v(0) = \pi(v)$ and $\dot{\gamma}_v(0) = v$. It follows $\dot{\gamma}_v(t) = \varphi^t(v)$ for any $v \in T^1M$ and $t \in \mathbb{R}$.
Let $K \subset T^1M$ be a compact φ^t-invariant set. It is well-known that there exists an equivariant splitting of the tangent bundle $T_K T^1 M$ with a one-dimensional subbundle E^0 tangential to the flow lines and a $(2n-2)$-dimensional subbundle transversal to the flow lines. Assume that all 2-planes Π which are tangent to the geodesic have strictly negative sectional curvature $S(\Pi)$. Since M is smooth and K is compact there are positive constants k_1 and k_2 with $-k_1^2 \le S(\Pi) \le -k_2^2$ for all 2-planes Π tangent to the geodesic in a point of $\pi(K)$. The $(2n-2)$-dimensional subbundle $T_K T^1 M$ which is transversal to the flow can equivariantly be splitted into a stable subbundle E^+ and an unstable subbundle E^- such that for any $u \in K$ and $t \in \mathbb{R}$ the relations

$$\frac{k_2}{k_1}\|\xi\| e^{-k_1 t} \le \|d_u \varphi^t \xi\| \le \frac{k_1}{k_2}\|\xi\| e^{-k_2 t} \quad \text{for any } \xi \in E_u^+$$

and

$$\frac{k_1}{k_2}\|\xi\| e^{k_1 t} \ge \|d_u \varphi^t \xi\| \ge \frac{k_2}{k_1}\|\xi\| e^{k_2 t} \quad \text{for any } \xi \in E_u^-$$

hold. It follows that the global Lyapunov exponents of the flow $\{\varphi^t\}$ on K with respect to the splitting $E^- \oplus E^+ \oplus E^0$ satisfy for $i = 1, \dots, 2n-2$ the inequalities $-k_1 \le \nu_i \le -k_2$.
If $(n-1)k_2 > h_{\mathrm{top}}(\varphi^1{}_{|K})$ there is a natural number $d_0 < 2n-1$ with $2h_{\mathrm{top}}(\varphi^1{}_{|K}) + \nu_1 + \ldots + \nu_{d_0} + \nu_{d_0+1} < 0$. Using now Corollary 33 we get the dimension estimate

$$\begin{aligned}
\dim_{\mathrm{H}} K &\le d_0 + 1 + \frac{2h_{\mathrm{top}}(\varphi^1{}_{|K}) + \nu_1 + \ldots + \nu_{d_0}}{|\nu_{d_0+1}|} \\
&\le d_0 + 1 + \frac{2h_{\mathrm{top}}(\varphi^1{}_{|K}) - d_0 k_2}{k_1} \le 1 + \frac{2h_{\mathrm{top}}(\varphi^1{}_{|K})}{k_1} + d_0\left(1 - \frac{k_2}{k_1}\right),
\end{aligned}$$

provided that $(n-1)k_2 < h_{\mathrm{top}}(\varphi^1_{|K})$ holds.

8 Upper Hausdorff Dimension Estimates for Non-Injective Maps

Whereas the majority of the previous sections were concerned with dimension estimates for injective dynamical systems here we want to consider time-discrete dynamical systems generated by a non-injective map. In this case the conditions of the Douady-Oesterlé dimension estimate of Theorem 4 can be weakened using the cardinality of the pre-image sets, described by the multiplicity function (see for instance [13]). For a map $\varphi : M_1 \to M_2$ between two arbitrary sets M_1 and M_2 the *multiplicity function* $N(\varphi, K, u)$ of φ with respect to a set $K \subset M_1$ in a point $u \in M_2$ is defined as the cardinality of the set $\{v \in K | \varphi(v) = u\}$. The following theorems are proved by A.Franz ([17]).

Theorem 35. *Let M be a n-dimensional Riemannian C^m-manifold, $U \subset M$ an open set, $\varphi : U \to M$ a C^1-map and $K \subset U$ a compact φ-invariant set with $|\det(d_u\varphi)| > 0$ for all $u \in K$. If there exist numbers $d \in (0, n]$ and $\nu \in (0, 1)$ with*

$$\omega_d(d_u\varphi)^{\frac{1}{d}} \le \nu N(\varphi, K, \varphi(u))^{\frac{1}{n}} \quad \text{for all } u \in K,$$

then $\dim_{\mathrm{H}} K \le d$.

Note that the condition of this theorem contains the singular-value function ω_d with an exponent $\frac{1}{d}$, since in contrast to ω_d the function $\omega_d^{1/d}$, as a function of d, is always monotone decreasing in the whole interval $(0, n]$.
The following theorem uses the local inverse maps, therefore a few additional assumptions are needed ($\mathrm{cl}(\cdot)$ denotes the closure of a set). The notation $\overline{\omega}_d$ refers to the inverse singular-value function defined as the product not of the largest but of the smallest d singular values.

Theorem 36. *Let M be a n-dimensional Riemannian C^m-manifold, $U \subset M$ an open set, $\varphi : U \to M$ a C^1-map and $K \subset U$ a compact φ-invariant set and $\widetilde{U} \supset K$ an open set with $\mathrm{cl}(\widetilde{U}) \subset U$ compact, $|\det(d_u\varphi)| > 0$ for all $u \in \mathrm{cl}(\widetilde{U})$ and $\varphi^{-1}(\widetilde{U}) \subset \widetilde{U}$. If there exist numbers $d \in (0, n]$ and $\nu \in (0, 1)$ with*

$$\nu\, \overline{\omega}_d(d_u\varphi) \ge N(\varphi, \widetilde{U}, \varphi(u)) \quad \text{for all } u \in K,$$

then $\dim_{\mathrm{H}} K \le d$.

The proof of the last two theorems is based on transformation formulas for Hausdorff outer integrals (see [17]), which have, applied to the characteristic function of a compact set, the same critical value as the outer Hausdorff-d-measure and can therefore be used in a similar way to obtain dimension estimates.

Example 37. Both theorems of this section can be applied to a wide class of non-invertible maps and to iterated function systems. Some classes of injective maps like horseshoe maps can be split into partial non-invertible maps, or the state space (for instance for Belykh maps ([1])) can be factorized such that the resulting non-injectivity of the map can be used to obtain dimension bounds for an invariant set. A well known class of non-injective maps are polynomials in the complex plane. We want to illustrate the dimension estimate by Theorem 36 for Julia sets of quadratic polynomials $\varphi : \mathbb{C} \to \mathbb{C}$ in the form

$$\varphi(z) = z^2 + c,$$

where $c \in \mathbb{C}$ is a parameter. The Julia set K is the closure of the set of repelling periodic points (see for instance [11,52]). This set is compact, non-empty and invariant under φ. For different parameters c this set has different shapes. Therefore it is interesting to estimate the Hausdorff dimension of this set.

Let us consider $\mathbb{C}$ as two-dimensional Riemannian C^∞-manifold with a constant metric tensor giving the metric

$$\varrho(z_1, z_2) = |z_1 - z_2| = \sqrt{(\mathrm{Re} z_1 - \mathrm{Re} z_2)^2 + (\mathrm{Im} z_1 - \mathrm{Im} z_2)^2}\ .$$

Under this metric the singular values of $d_z\varphi$ are

$$\sigma_1(d_z\varphi) = \sigma_2(d_z\varphi) = 2\sqrt{(\mathrm{Re} z)^2 + (\mathrm{Im} z)^2} = 2|z|.$$

Thus for $d \in (0, 2]$ and any $z \in \mathbb{C}$ we have $\omega_d(d_z\varphi) = \overline{\omega}_d(d_z\varphi) = (2|z|)^d$. It can be shown (see [11]) that the absolute value of the points of K can be estimated by

$$\begin{aligned} \frac{1}{2} + \sqrt{\frac{1}{4} - |c|} < |z| < \frac{1}{2} + \sqrt{\frac{1}{4} + |c|} \quad &\text{for } |c| < \frac{1}{4}, \\ \sqrt{|c| - \sqrt{|2c|}} \le |z| \le \sqrt{|c| + \sqrt{|2c|}} \quad &\text{for } |c| > 2. \end{aligned}$$

The multiplicity function of φ with respect to K and any open neighbourhood $\widetilde{U}$ of K satisfies $N(\varphi, K, \varphi(z)) = N(\varphi, \widetilde{U}, \varphi(z)) = 2$ for all $z \in K$ under the condition, that the origin is not contained in K. Thus the condition of Theorem 36 has the form $(2|z|)^d > 2$ for all $z \in K$ which is satisfied if

$$\begin{aligned} \left(1 + \sqrt{1 - 4|c|}\right)^d > 2 \quad &\text{for } |c| < \frac{1}{4}, \\ 2\sqrt{|c| - \sqrt{|2c|}} > 2 \qquad &\text{for } |c| > 2. \end{aligned}$$

Thus Theorem 36 gives, similar to [11], the dimension estimate

$$\dim_{\mathrm{H}} K \le \begin{cases} \frac{\ln 2}{\ln(1 + \sqrt{1 - 4|c|})} & \text{for } |c| < \frac{\sqrt{2} - 1}{2}, \\ \frac{\ln 2}{\ln\left(2\sqrt{|c| - \sqrt{|2c|}}\right)} & \text{for } |c| > \frac{3}{2} + \sqrt{2}, \end{cases}$$

In the case $|c| > \frac{3}{2} + \sqrt{2}$ results of [12] can be applied as well and yield the same dimension bound.

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