The Orbit Space Method: Theory and Application

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1 Introduction

If a system does not change its structure, while it is exposed to some transformations such as translations, rotations, reflections, or even more complicated transformations, it possesses symmetry. This is a feature of a host of phenomena. A good mathematical description of them must take the symmetry into account. Further symmetry may come about by simplifications. Often, deep statements are possible just by symmetry arguments. On the other hand, the presence of symmetry helps to analyze a given system. For example, one can find new solutions knowing a special one. Or, the symmetry may exclude some situations or force some phenomena to appear, which usually do not occur in generic nonsymmetrical systems.

We consider dynamical systems with symmetry being given by a group action on the phase space. Our aim is to use the symmetry to simplify the system in order to obtain information about its dynamical behavior, e.g. about bifurcation and stability properties. To this end, we propose to use the orbit space method which factors out the symmetry as follows. Due to symmetry, the dynamics of a system maps each (group) orbit onto an orbit. Therefore, the dynamics of such a system can be viewed as dynamics on the set of orbits, i.e., on the so-called orbit space (reduced dynamics). This projection reduces the complexity of dynamical objects. For example, a dynamically invariant set which is contained in a group orbit reduces to an equilibrium on the orbit space; therefore, its points are called relative equilibria. Similarly, relative periodic points are projected onto periodic points.

By means of this projection, the dynamics along any orbit is faded out, and in turn the directions transversal to the orbits are emphasized. In fact, all trajectories meeting one fixed orbit are projected onto the same trajectory on the orbit space, and show the same behavior. Modulo the symmetry, they cannot be distinguished. Thus, it seems reasonable to investigate them in their entirety. This is the basic idea of the orbit space method. For example, the stability of such a collection of trajectories as a union, i.e., orbital stability which is equivalent to stability on the orbit space, appears to be the appropriate notion of stability in this context. Throughout this paper, stability refers to this notion.

By the way, the term 'trajectory' is to be understood in the dynamical sense here and subsequently, while 'orbit' will exclusively be used in the context of group actions.

To handle the orbit space, it is convenient to embed it into some vector space. In finite dimensions, the embedding can be performed by a Hilbert basis; this is a set of generators of the algebra of polynomials on the phase space that are invariant under the action of the symmetry group. For compact or even reductive groups, the existence of a Hilbert basis is guaranteed by a famous theorem due to Hilbert and Weyl. Here embedding means to introduce coordinates which are invariant with respect to the group action. The new coordinates are nonlinear. Thus, degeneracies of the dynamics forced by the symmetry are converted into singularities of the orbit space. We are not going to exclude this kind of singularities later on. Rather, a key feature of this contribution is to show how to apply the orbit space method at such singularities (singular reduction).

There are several papers using orbit space reduction for dynamical systems with symmetry, e.g. [2], [25], [7], [8], [9], [26], [36], [35]. See also the contribution by F. Guyard and R. Lauterbach to the present volume. Also, there is an extensive literature about more sophisticated versions for particular systems such as mechanical systems (see, for example, [1] and the references therein). However, this is far beyond the theory discussed here. For the approach of restriction to fixed point spaces of isotropy subgroups we refer to [16,17].

In the present paper, we give a theoretical justification of the orbit space method for general dynamical systems with symmetry, and outline fundamental techniques for its application. Most topics in this paper concern both continuous and discrete dynamics. In Sect. 2, we start with a suitable description of the symmetry, introduce the orbit space, and develop the concept of reduction to the orbit space. An extension ([34]) of the fundamental theorem of Schwarz ([37]) allows to consider not only smooth, but also finitely differentiable systems, which arise, for instance, from a center manifold reduction. In this sense, infinite-dimensional systems are included in the theory developed here. In fact, our main example will be an application to a system of Kuramoto-Sivashinsky type of partial differential equations which is of that nature (Sect. 7).

The main purpose of the orbit space method is to draw conclusions from the reduced dynamics to the dynamics of the original system. Here we concentrate on dynamical features such as bifurcation and stability properties that can be characterized by eigenvalues (exponents) of appropriately constructed linearized systems. For example, since the orbit space will be embedded into a vector space, we extend the reduced dynamics to this vector space and consider the linearization of the extended system at points of the orbit space. In particular, this linearization is well defined at singular points. Since the process of reduction uses nonlinear coordinates, at first glance it is by no means obvious how eigenvalues (exponents) on the orbit space relate to corresponding eigenvalues (exponents) on the original phase space, and vice versa. Of course, only the modes in directions transversal to group orbits in the original phase space play a role in this context. Moreover, since the orbit space will be embedded into a vector space of higher dimension, in general, not all eigenvalues (exponents) inside this vector space affect the dynamics on the orbit space. We are going to address both these issues for equilibria of maximal symmetry (isotropy) in Sect. 3, for general relative equilibria in Sect. 4, for relative periodic points in Sect. 5, and finially, for even more general trajectories in Sect. 6. In all cases it turns out that there are certain algebraic expressions for the eigenvalues (exponents) on the orbit space in terms of the ones on the original phase space depending on the representation of the symmetry group. If the dynamics is given by a vector field, then eigenvalues of the original system appear as linear combinations over $\mathbb{N}_0$ on the orbit space (cf. [23]). The same is true for exponents. For discrete systems, products of eigenvalues of the original system appear on the orbit space. In particular, for compact symmetry groups the relevant eigenvalues (exponents) on the orbit space are sufficient to determine the spectral (orbital) stability and instability of trajectories of the original system.

Recall from above that the orbit space is not a smooth manifold, in general. Rather, it is a semialgebraic variety. To determine the relevant eigenvalues on the orbit space, we use a concept of tangent space (tangent bundle) which is adopted from algebraic geometry, and restrict the linearized system to that. The tangent space at equilibria of maximal symmetry always covers the whole vector space the orbit space is embedded in, if the Hilbert basis is minimal. Hence, at such points the computations are easier than at general points.

To deal with general relative equilibria we use the slice theorem. Together with a splitting of the dynamics into parts tangent to the orbits and transversal to them due to M. Field [14] and M. Krupa [24] (cf. also [12]), it allows to reduce an arbitrary point to a point with maximal symmetry. Furthermore, it provides a suitable stratification for the corresponding representation space. This partition into sets of 'equal' symmetry is preserved during the reduction process.

Our results for relative equilibria also apply to relative periodic points. To show this we construct an invariant hypersurface transversal to the flow and a first return map on this hypersurface (cf. [13]). This map is group invariant and possesses a relative equilibrium corresponding to the relative periodic point. It reduces to a Poincaré map on the orbit space. In [31] this result is generalized to Lyapunov exponents. In Sect. 6, we just indicate the main differences between the cases of eigenvalues and Lyapunov exponents. For details concerning the latter, we refer the interested reader to [31].

We point out that the concept of a tangent space as indicated above is not yet sophisticated enough to get rid of any spurious eigenvalue on the orbit space. For instance, at a point corresponding to a hyperbolic dynamically invariant orbit of saddle type the linearization of the reduced system restricted to the tangent space may have critical eigenvalues besides stable and unstable ones. In other words, hyperbolicity on the original phase space does not necessarily imply hyperbolicity on the orbit space in that context. To accomplish that this implication is true, the concept of a tangent cone which is a subset of the tangent space has been introduced by M. Kœnig [23]. In Sect. 8, we invoke this concept to refine our theory as to reproduce all eigenvalues transverse to the group orbits in the original phase space from the eigenvalues on the orbit space. Thus, we establish a kind of converse to the results of the previous sections.

For a Hartman-Grobman type of linearization theory on the orbit space, see [10].

Finally, Sect. 9 is devoted to a discussion of the case of reductive symmetry groups. These are groups whose finite-dimensional representations are completely reducible. They are not compact, in general. Examples show that the orbit space method may fail in that case.

2 Reduction of symmetric systems

First we introduce some notation. Let Γ be a compact Lie group acting orthogonally on the real vector space $V = \mathbb{R}^m$. Let $P : V \to V$ be a Γ-equivariant map. This means that P commutes with the group action: $\forall x \in V, \gamma \in \Gamma : P(\gamma x) = \gamma P(x)$. Therefore, the image of each point γx of the orbit Γx is known, if $P(x)$ is known. Let F be a Γ-equivariant vector field on V. Then, due to the uniqueness, the flow Φ_t of F is a Γ-equivariant map $V \to V$ for each t. Assume P and F, and therefore Φ_t, to be qn-times continuously differentiable, where the integers q and n are to be fixed later.

The trajectories of all points of an orbit are conjugated by the symmetry group. Moreover, an orbit Γx is mapped onto the orbit $\Gamma P(x) = P(\Gamma x)$ or $\Gamma \Phi_t(x)$ by P or Φ_t, respectively. Thus, it is reasonable to regard the dynamics as a system on the set $\{\Gamma x \mid x \in V\}$ of orbits, called the *orbit space*. It is identified with the quotient V/Γ. The canonical projection $\pi_\Gamma : \begin{cases} V \longrightarrow V/\Gamma \\ x \longmapsto \Gamma x \end{cases}$ induces a topology on V/Γ.

The map $\pi_\Gamma \circ P$ is Γ-invariant, i.e., constant on all orbits. Therefore, there exists a map $\overline{P} : V/\Gamma \to V/\Gamma$ with

$$\overline{P} \circ \pi_\Gamma = \pi_\Gamma \circ P \ .$$

In this way, the symmetry is divided out of P, and a possibly simpler map comes up. If Γx is a dynamically invariant orbit, meaning that no trajectory of P leaves the set Γx, then $\Gamma x = \pi_\Gamma(x)$ is an equilibrium of $\overline{P}$. We use the word ‘equilibrium’ also for a fixed point of a map to distinguish from a ‘fixed point’ of the group action. In the sense of orbital stability, the stability of Γx

with respect to P agrees with the stability of $\pi_\Gamma(x)$ with respect to $\overline{P}$. In a similar way, other dynamical objects corresponding to P reduce to simpler objects corresponding to $\overline{P}$.

The embedded orbit space To handle the orbit space and the reduced dynamics, it is convenient to embed the orbit space into a vector space. To do so, we introduce a kind of invariant coordinates.

The action of Γ on V induces an action on the functions $f : V \to N$, where N is some set:

$$\gamma f : x \longmapsto f(\gamma^{-1}x) \ .$$

The fixed point space of this action contains exactly the Γ-invariant functions. As usual, we mark the fixed point space by a superscript Γ, e.g. $\mathbb{R}[V]^\Gamma$ is the $\mathbb{R}$-algebra of Γ-invariant polynomials. A well-known theorem by Hilbert and Weyl states that this algebra is finitely generated ([42]). A set of generators $\rho_1, \ldots, \rho_k$ is called a *Hilbert basis*. Each Γ-invariant polynomial f can be written as a polynomial g in $\rho_1, \ldots, \rho_k$:

$$f(x) = g(\rho_1(x), \ldots, \rho_k(x)) \ .$$

To simplify notations, we define the *Hilbert map* $\rho(x) = (\rho_1(x), \ldots, \rho_k(x))$. (It is quite difficult to compute a Hilbert basis for a given representation, but there exist algorithms for finite groups (see Sturmfels [40], Kemper [22]) and for compact and reductive groups (see Gatermann [15] and Derksen [11]).)

The polynomial $R : x \mapsto \langle x, x\rangle$ is Γ-invariant, due to the orthogonality of the action. Therefore, there is a polynomial g with $R = g \circ \rho$. Let $K \subset \mathbb{R}^k$ be a compact set and L a compact interval containing $g(K)$. Then $R^{-1}(L)$ is closed and bounded, and contains $\rho^{-1}(K) \subset R^{-1}(g(K))$. Thus, the preimage $\rho^{-1}(K)$ itself is compact. This means that a Hilbert map of an orthogonal representation is proper. As a consequence, ρ is closed and even open as a map from V into $Y := \rho(V)$.

Since the action of Γ on $\mathbb{R}[V]$ preserves the degree, we can choose homogeneous generators of $\mathbb{R}[V]^\Gamma$. In this case, we say the Hilbert basis and the Hilbert map are *homogeneous*. A homogeneous Hilbert basis and the corresponding Hilbert map are called *minimal*, if each proper subset of the basis does not generate $\mathbb{R}[V]^\Gamma$. In this paper ρ is assumed to be a homogeneous Hilbert map with image $Y \subset \mathbb{R}^k$.

Assuming a decomposition $V = V_1 \oplus \cdots \oplus V_l$ with coordinates $x = (x_1, \ldots, x_l)$ we can generalize the notion of degree in the following way. For a multiindex $\beta \in \mathbb{N}_0^l$, denote by $\mathbb{R}[V]_\beta$ the subspace of polynomials which are homogeneous of degree β_i in x_i, $i = 1, \ldots, l$. These polynomials are called *multihomogeneous*, and β is their *multidegree*. If the spaces V_i are Γ-invariant, then $\mathbb{R}[V]_\beta$ is Γ-invariant for all β, and a multihomogeneous Hilbert basis exists.

Remark 1. The number of elements of a minimal Hilbert basis and their (multi-)degrees are unique up to permutation.

Using the Weierstraß approximation theorem it is easy to see that a Hilbert map separates the Γ-orbits, i.e., the images $\rho(x)$ and $\rho(y)$ differ for $x,\, y \in V$ with $x \notin \Gamma y$. Hence, the map $\overline{\rho} : V/\Gamma \to \mathbb{R}^k$ with $\overline{\rho} \circ \pi_\Gamma = \rho$ is one-to-one. Moreover, it is a homeomorphism. Therefore, we identify the orbit space V/Γ with $Y = \rho(V)$, and call Y the *(embedded) orbit space.*

A relation of ρ is a polynomial $r \in \mathbb{R}[\mathbb{R}^k]$ with $r \circ \rho = 0$. The relations form an ideal I in $\mathbb{R}[\mathbb{R}^k]$. If a given Γ-invariant polynomial f is represented by $g \in \mathbb{R}[\mathbb{R}^k]$, then all polynomials $h \in g + I$ represent f by $f = h \circ \rho$. Thus, in contrast to $g|_Y$, g is not unique, if I is nontrivial.

The variety $Z := \{y \in \mathbb{R}^k \mid \forall r \in I : r(y) = 0\}$ associated with this ideal is the smallest variety containing Y. The subset Y of Z is described by some polynomial inequalities obtained as follows. Define the $k \times k$-matrix $G(x)$ by $G_{i,j}(x) := \operatorname{grad} \rho_i(x) \cdot \operatorname{grad} \rho_j(x)$. Then G is Γ-invariant, and there exists a map $H : \mathbb{R}^k \to \mathbb{R}^{k \times k}$ with $H(\rho(x)) = G(x)$. If $y \in Z$ lies in Y, then $H(y)$ is positive semi-definite. It is proved in [30] that the converse is also valid. Therefore, the inequalities which describe the definiteness of $H(y)$ determine the shape of $Y \subset Z$. Thus, Y is a semialgebraic set.

The differentiability of the reduced system Since Γ-invariant polynomials can be represented on the orbit space by polynomials, one may ask, whether a differentiable Γ-invariant function can be represented by a differentiable function. For example, the even (i.e., $\mathbb{Z}_2$-invariant) function $f : x \mapsto |x|^3$ is C^2, but all the functions g with

$$f = g \circ \rho \,, \tag{1}$$

where $\rho(x) = x^2$, are only C^1, since $g : y \to |y|^{3/2}$ on $Y = \mathbb{R}_0^+$. However, there is a C^1-function g satisfying (1). In general, a loss of differentiability is expected, but this loss is controllable:

Proposition 2 ([37,27,34]). *There exists an integer q such that for each $n = 0, \ldots, \infty$ a continuous map*

$$\mathcal{J} : C^{qn}(V)^\Gamma \longrightarrow C^n(\mathbb{R}^k)$$

exists satisfying $\mathcal{J}(f) \circ \rho = f$ for $f \in C^{qn}(V)^\Gamma$.
The Taylor polynomial of degree n of $\mathcal{J}(f)$ depends only on the Taylor polynomial of degree qn of f.

The integer q depends only on the representation, and is at least the degree of a minimal Hilbert basis.

We use this result to find differentiable representations of P and F on the orbit space. $\rho \circ P$ is a Γ-invariant C^{qn}-map $V \to \mathbb{R}^k$. We apply the proposition to each component, and get a C^n-map $Q : \mathbb{R}^k \to \mathbb{R}^k$ such that the diagram

$$\begin{array}{ccc} V & \xrightarrow{P} & V \\ {\scriptstyle\rho}\big\downarrow & & \big\downarrow{\scriptstyle\rho} \\ \mathbb{R}^k & \xrightarrow{Q} & \mathbb{R}^k \end{array} \tag{2}$$

commutes. Denote the space of C^n-maps from V to W by $C^n(V, W)$, and indicate the subspace of equivariant maps by a subscript Γ: $C^n_\Gamma(V, W)$. Then there is a continuous map

$$\mathcal{J}_{\mathsf{m}} : C^{qn}_\Gamma(V, V) \longrightarrow C^n(\mathbb{R}^k, \mathbb{R}^k)$$

which satisfies $\mathcal{J}_{\mathsf{m}}(P) \circ \rho = \rho \circ P$ for all $P \in C^{qn}_\Gamma(V)$.

Also vector fields can be reduced by factoring out the symmetry. The space of vector fields of class C^n on V is denoted by $\mathrm{Vec}^n(V)$. The subspace of Γ-equivariant vector fields is denoted by $\mathrm{Vec}^n_\Gamma(V)$. There is a continuous and linear map

$$\mathcal{J}_{\mathsf{v}} : \mathrm{Vec}^{qn}_\Gamma(V) \longrightarrow \mathrm{Vec}^n(\mathbb{R}^k)$$

with $\mathcal{J}_{\mathsf{v}}(F) \circ \rho(x) = D\rho(x)F(x)$ for all $F \in \mathrm{Vec}^{qn}_\Gamma(V)$. Denote the flow of $\mathcal{J}_{\mathsf{v}}(F)$ by Ψ_t, then we obtain $\rho \circ \Phi_t = \Psi_t \circ \rho$.

This approach to symmetric systems works on Γ-manifolds, too. Let M be a smooth manifold without boundary, and suppose there is a smooth action of Γ on M. The only further condition on M is that the action has at most finitely many orbit types. Then there is a Γ-equivariant embedding of M into a representation space V of Γ. Since a Hilbert map ρ is proper, an open set W exists such that $\rho(M) = \rho(V) \cap W$. Now $\rho : V \to \mathbb{R}^k$ may be replaced by $\rho : M \to W$ in the following sections.

Reducing linear maps Even if the Γ-equivariant map P is linear, the reduced map Q need not to be linear. Since relations can be added to Q, it is interesting to ask, whether one can choose a linear Q obeying (2). We can restrict our investigations to the case of minimal Hilbert bases.

Consider the effect of P as an action on $\mathbb{R}[V]^\Gamma$. The maximal ideal J of polynomials vanishing at 0 is invariant with respect to P. Each Hilbert basis generates J. If the Hilbert basis is minimal, then it is projected by the canonical surjection to a vector space basis of J/J^2. Therefore, a minimal Hilbert basis is a vector space basis of a complement C of J^2 in J, and vice versa.

Suppose C is invariant with respect to P. Then $\rho_i \circ P$ is in C, and there is a linear function Q_i with $\rho_i \circ P = Q_i \circ \rho$. The question arises whether a P-invariant complement of J^2 exists.

Example 3. Consider the natural action of $\mathbb{Z}_4$ on $\mathbb{R}^2$. A Hilbert basis is given by $u(x, y) = x^2+y^2$, $v(x, y) = x^2y^2$, $w(x, y) = x^3y-xy^3$. There is the relation

$w^2 = v(u^2 - 4v)$. A $\mathbb{Z}_4$-equivariant linear map has the form $A := \begin{pmatrix} \lambda & -\mu \\ \mu & \lambda \end{pmatrix}$. We have:

$$\begin{aligned} u(A(x,y)) &= u(\lambda,\mu)u(x,y) \qquad u(A(x,y))^2 = u(\lambda,\mu)^2 u(x,y)^2 \\ v(A(x,y)) &= v(\lambda,\mu)u(x,y)^2 + 2w(\lambda,\mu)w(x,y) + (u(\lambda,\mu)^2 - 8v(\lambda,\mu))v(x,y) \\ w(A(x,y)) &= (u(\lambda,\mu)^2 - 8v(\lambda,\mu))w(x,y) + w(\lambda,\mu)(u(x,y)^2 - 8v(x,y)) \end{aligned}$$

Take $v'(A(x,y)) = u(x,y)^2 - 8v(x,y) = x^4 - 6x^2y^2 + y^4$ instead of v ($v'^2 + 16w^2 = u^4$). Then

$$\begin{aligned} v'(A(x,y)) &= v'(\lambda,\mu)v'(x,y) - 16w(\lambda,\mu)w(x,y) \\ w(A(x,y)) &= w(\lambda,\mu)w(x,y) + v'(\lambda,\mu)w(x,y) \ . \end{aligned}$$

Thus, all linear maps can be reduced to linear maps on the orbit space, using the Hilbert basis u, v', and w. This is also true for vector fields:

$$\begin{aligned} \operatorname{grad} u \cdot A &= 2\lambda u & \operatorname{grad} v \cdot A &= 4\lambda v + 2\mu w \\ \operatorname{grad} w \cdot A &= \mu(u^2 - 8v) + 4\lambda w = \mu v' + 4\lambda w & \operatorname{grad} v' \cdot A &= 4\lambda v' - 16\mu w \end{aligned}$$

Proposition 4. *There is a Hilbert basis such that for all Γ-equivariant automorphisms P a linear map Q on $\mathbb{R}^k$ satisfying (2) exists.*

Proof. $\Delta := \mathrm{GL}(V)^\Gamma$ is the group of Γ-equivariant automorphisms of V. It is the centralizer of Γ in $\mathrm{GL}(V)$. If the i-th isotypical component of V has type $\mathbb{K}_i$ and multiplicity n_i, then $\Delta \cong \bigotimes_i \mathbb{K}_i \otimes_{\mathbb{R}} \mathrm{GL}(\mathbb{R}^{n_i})$ is a reductive group. Thus, the representation of Δ on $J \cap \mathbb{R}[V]^\Gamma_{\leq d}$, where $d := \max_i \deg \rho_i$, is completely reducible, and the invariant space $J^2 \cap \mathbb{R}[V]^\Gamma_{\leq d}$ has an invariant complement. □

The same is valid for vector fields, see [23].

3 Eigenvalues at equilibria of maximal isotropy

We consider a Γ-equivariant map $P : V \to V$ which has an equilibrium invariant under the the full symmetry group Γ. Without loss of generality we assume this equilibrium to be 0. The eigenvalues of the linearization $DP(0)$ indicate the stability of the equilibrium, if they lie inside the unit circle. The reduced map Q has an equilibrium at $0 = \rho(0)$. The question arises, how the eigenvalues of the linearizations $DP(0)$ and $DQ(0)$ are related. In particular, can the stability of $0 = P(0)$ be read off from the eigenvalues of $DQ(0)$?

If a relation $r \in I$ with non-vanishing linear terms exists, then $DQ(0)$ is not unique. Moreover, then we can choose Q in such a way that an arbitrarily given number λ is an eigenvalue of $DQ(0)$. However, the following straight forward lemma provides a condition for uniqueness.

Lemma 5. *A homogeneous Hilbert basis is minimal, if and only if the linear part* $\operatorname{grad} r(0)$ *of all relations* $r \in I$ *vanishes. In this case, the linear part of* $Q \in C^n(\mathbb{R}^k, \mathbb{R}^k)$ *satisfying (2) depends uniquely on* $P \in C_\Gamma^{qn}(V, V)$.

Theorem 6 ([32]). *Let* P *and* Q *be as in (2). Suppose* $P(0) = 0$, *and assume the Hilbert map* ρ *to be minimal. Then the eigenvalues of* $DQ(0)$ *are products of eigenvalues of* $DP(0)$, *a factor may occur several times in each product.*

Proof. Assume P to be polynomial. The map $Q = \mathcal{J}_{\mathsf{m}}(P) : \mathbb{R}^k \to \mathbb{R}^k$ satisfies (2). Decompose P into a sum $P = P_\ell + P_{\mathrm{r}}$, where P_ℓ contains the linear terms of P, and P_{r} consists of the terms with degree greater than 1. Since the action preserves the degree, P_ℓ and P_{r} are Γ-equivariant.

Suppose that the Hilbert basis is homogeneous and ordered by increasing degree. Then $DQ(0)$ has the block form

$$\begin{pmatrix} A_1 & * & \cdots & * \\ 0 & A_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & A_j \end{pmatrix}, \tag{3}$$

where the matrix A_1 belongs to the ρ_i's with the lowest degree, and so on. The vanishing entries in the lower triangle result from the increasing degrees of the elements of the Hilbert basis: Since ρ is minimal, the terms of $\rho_i \circ P$ have at least degree $\deg \rho_i$. Here we use the last assertion of Proposition 2. A similar argument yields that the entries above the diagonal depend only on P_{r}, and P_{r} has no effect on the A_i's. Therefore, we can ignore P_{r} without changing the eigenvalues of $DQ(0)$.

First we assume $P_\ell = P$ to be semisimple. Let $\lambda_1, \ldots, \lambda_l$ be the eigenvalues of P_ℓ. Since P_ℓ is Γ-equivariant, the corresponding eigenspaces V_i are Γ-invariant: If $v \in V_i$, then $P_\ell(\gamma v) = \gamma P_\ell(v) = \gamma \lambda_i v = \lambda_i \gamma v$, thus $\gamma v \in V_i$. We have, changing the coordinates in V,

$$P = \begin{pmatrix} \lambda_1 \operatorname{id}_{V_1} & & 0 \\ & \ddots & \\ 0 & & \lambda_l \operatorname{id}_{V_l} \end{pmatrix}.$$

Denote the coordinates in V_i by x_i, and let $\sigma_1, \ldots, \sigma_k$ be a minimal Hilbert basis consisting of multihomogeneous polynomials. If there are complex eigenvalues, it is understood that we pass to the complexification $V_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} V$ of the representation space V. However, a Hilbert basis on V is a real Hilbert basis on $V_{\mathbb{C}}$, and vice versa.

According to the classical theory, there is a polynomial map $Q' : \mathbb{R}^k \to \mathbb{R}^k$ with $Q' \circ \sigma = \sigma \circ P$. As a consequence of the following lemma, $DQ'(0)$ and $DQ(0)$ have the same spectrum.

Lemma 7. *The spectrum of the reduced map is independent of the choice of the minimal Hilbert basis. Passing to another minimal Hilbert basis means a polynomial change of coordinates, and conjugating the differentials.*

Proof. Let ρ and σ be minimal Hilbert maps. Obviously, polynomial maps $\Lambda, \mathrm{M} : \mathbb{R}^k \to \mathbb{R}^k$ with $\rho = \Lambda \circ \sigma$ and $\sigma = \mathrm{M} \circ \rho$ exist. From the minimality of the Hilbert bases, $D\Lambda(0)^{-1} = D\mathrm{M}(0)$ follows. Here, we use the fact that the differentials are uniquely determined by those relations (cf. Lemma 5).

Let Q' be the reduced map with respect to σ: $\sigma \circ P = Q' \circ \sigma$. Then we have $\Lambda \circ Q' \circ \mathrm{M} = Q$ on Y, and $DQ(0) = D\Lambda(0) DQ'(0) D\Lambda(0)^{-1}$. □

Write

$$\sigma_j(x) = \sum_\alpha c_\alpha \prod_{i=1}^{l} x_i^{\alpha_i} ,$$

where the sum runs over the l-tuple of multiindices $\alpha_i \in N_0^{\dim V_i}$. According to our assumption, there exists a multiindex $\beta \in N_0^l$ such that $c_\alpha \neq 0$, only if $|\alpha_1| = \beta_1, \ldots, |\alpha_l| = \beta_l$. Now compute

$$Q'_j(\sigma(x)) = \sigma_j(P(x)) = \sum_\alpha c_\alpha \prod_{i=1}^{l} (\lambda_i x_i)^{\alpha_i} = \sigma_j(x) \prod_{i=1}^{l} \lambda_i^{|\alpha_i|} \tag{4}$$

Thus, Q' is linear and diagonal, and its eigenvalues are some products of eigenvalues of P. Since $DQ(0)$ and $Q' = DQ'(0)$ are conjugated, we have calculated the eigenvalues of $DQ(0)$ for a polynomial map $P : V \to V$ with semisimple linear part.

Now, take an arbitrary $P \in C_\Gamma^{qn}(V, V)$. Then there is a sequence of polynomials $P_i \in C_\Gamma^{qn}(V, V)$ with semisimple linear parts, which converges to P. Since $\widetilde{\mathcal{J}} : P \mapsto D\mathcal{J}_{\mathsf{m}}(P)(0) \in \mathbb{R}^{k \times k}$ is continuous, and the eigenvalues depend in some sense continuously on a matrix (see [21, II Thm. 5.14]), the claim holds for all maps P. □

Corollary 8. *In the situation of the last theorem, let λ be an eigenvalue of $DP(0)$, then $DQ(0)$ has the eigenvalue λ or $|\lambda|^2$.*

Proof. Resume the notations used in the proof of the theorem. Denote the eigenspace of λ by V_λ. Let $W \subset V_\lambda$ be an irreducible subrepresentation. If W is the trivial representation, then the coordinate σ_1 in $W \cong \mathbb{R}$ is Γ-invariant, and there is a minimal multihomogeneous Hilbert basis containing σ_1. According to (4), λ is an eigenvalue of $DQ(0)$.

Now suppose that W is nontrivial. If λ is real, then the eigenspace of λ contains a nontrivial (real) representation W'. This representation is orthogonal, hence the square σ_1 of the Euclidean norm on W' is Γ-invariant. Moreover, there are no linear invariants. Therefore, there is a multihomogeneous Hilbert basis containing σ_1. And according to (4), $\lambda^2 = |\lambda|^2$ is an eigenvalue of $DQ(0)$.

If λ is complex, then $\bar{\lambda}$ is also an eigenvalue, and the eigenspace $V_{\bar{\lambda}}$ contains the conjugated representation $\overline{W}$. If x_W and $x_{\overline{W}}$ are the coordinates of W and $\overline{W}$, respectively, then $\sigma_1(x) := \langle x_W, \mu x_{\overline{W}} \rangle$ is a Γ-invariant polynomial of degree 2, where $\mu : \overline{W} \to W$ is a anti-linear Γ-equivariant isomorphism. Since there are no linear invariants on W and $\overline{W}$, a minimal multihomogeneous Hilbert basis containing σ_1 exists. And again according to (4), $\lambda\bar{\lambda} = |\lambda|^2$ is an eigenvalue of $DQ(0)$. □

More details concerning the relation between the eigenvalues of $DP(0)$ and $DQ(0)$ will be given in Sect. 8. By Theorem 6 and Corollary 8 we conclude that the eigenvalues of $DP(0)$ lie inside the (closed) unit disc, if and only if all eigenvalues of $DQ(0)$ have this property. In particular, it follows by the principle of linearized stability that 0 is an asymptotically stable equilibrium of P, if all eigenvalues of $DQ(0)$ are strictly inside the unit disc; it is an unstable equilibrium, if at least one eigenvalue of $DQ(0)$ lies outside the closed unit disc. Also, if $DP(0)$ has a critical eigenvalue, then the same holds for $DQ(0)$. In general, the converse is not true. However, it is true provided that the spectrum of $DQ(0)$ is contained in the closed unit disc.

Remark 9. (Cf. Sect. 8) Let R be the subspace of $\mathbb{R}^k$ associated with the linear and quadratic components of ρ. Then $DQ(0)$ keeps R invariant. To determine the stability of the equilibrium 0, it suffices to check the eigenvalues of $DQ(0)|_R$.

Example 10. Let $\Gamma = \mathbb{Z}_4$ act freely on $\mathbb{R}^2$. A minimal Hilbert basis is given by $u(x,y) = x^2 + y^2$, $v(x,y) = x^2y^2$, and $w(x,y) = x^3y - xy^3$. A $\mathbb{Z}_4$-equivariant map P has the equilibrium $P(0) = 0$ and the linear part $\begin{pmatrix} \lambda & -\mu \\ \mu & \lambda \end{pmatrix}$. The reduced map Q has the differential $DQ(0) = \begin{pmatrix} \lambda^2+\mu^2 & * & * \\ 0 & \lambda^4 - 6\lambda^2\mu^2 + \mu^4 & 2\lambda^3\mu - 2\lambda\mu^3 \\ 0 & -8\lambda^3\mu - 8\lambda\mu^3 & \lambda^4 - 6\lambda^2\mu^2 + \mu^4 \end{pmatrix}$.
The eigenvalues of $DP(0)$ are $\lambda + \mathrm{i}\mu$ and $\lambda - \mathrm{i}\mu$, while $DQ(0)$ has the eigenvalues $\lambda^2 + \mu^2$, $(\lambda + \mathrm{i}\mu)^4$, and $(\lambda - \mathrm{i}\mu)^4$. The eigenvalue $\lambda^2 + \mu^2$, which corresponds to the quadratic invariant u, suffices to determine asymptotic stability or instability of the equilibrium $0 = P(0)$.

Example 11. Let $\Gamma = \mathbb{Z}_2$ act on $\mathbb{R}^2$ as $\pm\,\mathrm{id}$. A minimal Hilbert basis is given by $\rho_1(x,y) = x^2$, $\rho_2(x,y) = y^2$, and $\rho_3(x,y) = xy$. They have the relation $r(\rho_1,\rho_2,\rho_3) = \rho_1\rho_2 - \rho_3^2$. The orbit space is contained in the variety $r = 0$, and given by $\rho_1 \geq 0$.

Let P be a $\mathbb{Z}_2$-equivariant map. Then P has the equilibrium 0 and the linear part $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The eigenvalues of $DP(0)$ are $\lambda_{1,2} := \frac{a+d}{2} \pm \frac{1}{2}\sqrt{(a-d)^2 + 4bc}$.

The reduced map Q has the linear part $DQ(0) = \begin{pmatrix} a^2 & b^2 & 2ab \\ c^2 & d^2 & 2cd \\ ac & bd & ad+bc \end{pmatrix}$ with eigenvalues $\mu_1 = \lambda_1^2$, $\mu_2 = \lambda_2^2$ and $\mu_3 = \lambda_1\lambda_2$.

Observe, that if $\lambda_1 = \lambda_2^{-1}$ and $\lambda_1 > 1$, then the eigenvalue μ_3 of $DQ(0)$ lies on the unit circle, but no one of $DP(0)$ does. This phenomenon might occur, if P and hence Q have unstable eigenvalues.

Here we can use the eigenvectors corresponding to μ_3 in order to exclude that $DP(0)$ has a critical eigenvalue. An eigenvector of μ_3 is given by $v := (-b, c, \frac{a-d}{2})$. The line $\mathbb{R}v$ meets the orbit space nontrivially, if and only if the eigenvalues λ_1 and λ_2 coincide, i.e., $(a-d)^2 - 4bc = 0$. In this case, μ_3 is equal to μ_1 and μ_2. Thus, $DP(0)$ has a critical eigenvalue, if and only if μ_3 is critical and has an eigenvector in the orbit space. We show in Sect. 8 how to handle such situations more generally.

As a final conclusion from Theorem 6 we state the following corollary.

Corollary 12. *If P is a local diffeomorphism at 0, then the same holds true for Q.*

Proof. No eigenvalue of $DP(0)$ vanishes, hence all eigenvalues of $DQ(0)$ differ from zero. Therefore, $DQ(0)$ is a local diffeomorphism. □

Of course, analogous results are valid for vector fields (cf. [23,33]).

Theorem 13. *Let $F \in \mathrm{Vec}^q_\Gamma(V)$ and $G = \mathcal{J}_v(F)$. Suppose that $F(0) = 0$, and ρ is a minimal Hilbert map. Then the eigenvalue of $DG(0)$ are sums of the eigenvalues of $DF(0)$.*
If λ is an eigenvalue of $DF(0)$, then $DG(0)$ has λ or $\lambda + \bar{\lambda}$ in its spectrum.

4 Eigenvalues at a relative equilibrium

In this section the meaning of Γ, V, ρ, Y, P, and Q is as before. Next we study dynamically invariant orbits. Assume the orbit Γx to be invariant with respect to P or, equivalently, $P(x) \in \Gamma x$. Such a point x is called a *relative equilibrium*. Then the orbit Γx of x is an *invariant orbit*.

As mentioned above, an invariant orbit Γx reduces to an equilibrium $\rho(x)$ on the orbit space. Moreover, the stability of Γx is equivalent to the stability of $\rho(x)$ with respect to $Q|_Y$. Since Q does not know anything about the dynamical drift along the orbits, we cannot expect to find statements about the stability of the single trajectory of x using Q, but only for the whole orbit Γx as a set. This is a natural point of view, because all points of the orbit show exactly the same dynamical behavior.

We extend Theorem 6 to points x with submaximal symmetry, and exploit the eigenvalues of $DQ(\rho(x))$ to determine the stability of the invariant orbits Γx. For this we need to have informations about the local structure of the orbit space near $\rho(x)$. The slice theorem states that the orbit space locally looks like the origin of the orbit space of the isotropy group Γ_x. Furthermore, it is necessary to introduce a kind of linearization of P along an orbit. We begin by discussing the relevant concepts.

Local structure of the orbit space: The slice theorem An important concept in this context is the *isotropy subgroup* Γ_x *of a point* x. This group of all $\gamma \in \Gamma$ which fix x ($\Gamma_x := \{\gamma \in \Gamma \mid \gamma x = x\}$) describes the local symmetry.

In our simple situation tangent spaces inherit their inner products from the vector space V. If we work with manifolds, we use a Γ-invariant Riemannian metric, which exists always. The exponential map $\exp_x : T_x V \to V$ is a local Γ_x-equivariant diffeomorphism. In our case we can identify $T_x V$ with V, and $\exp_x$ is just the translation by x.

Let N_x be the orthogonal complement of $T_x \Gamma x$ in $T_x V$. N_x is the space of vectors normal to the orbit, and is called the *normal space* at x. The dimension of N_x depends on the dimension of the orbit, and is constant on orbits. The isotropy subgroup Γ_x acts naturally on the tangent space $T_x V$. This action emerges from taking the derivation of the action of Γ. Note that this action is orthogonal, too, and that the tangent space $T_x \Gamma x$ of the orbit and, therefore, N_x are Γ_x-invariant.

Since a group element γ maps $T_x \Gamma x$ onto $T_{\gamma x} \Gamma x$, we have $\gamma : N_x \to N_{\gamma x}$. For a small $\varepsilon > 0$ and $y \in \Gamma x$ we call the ε-neighborhood S_y of 0 in N_y a *slice at* y, which is Γ_y-invariant. We identify S_y with its image $\exp_y(S_y)$, and consider N_y as a subspace of V.

If $\varepsilon > 0$ is small enough, an ε-neighborhood U_x of the orbit Γx carries the structure of a twisted product or of a fiber bundle with fiber S_x over the orbit. The isotropy group Γ_x acts on the product $\Gamma \times S_x$ by

$$(\delta, (\gamma, v)) \longmapsto (\gamma\delta^{-1}, \delta v) \ .$$

Let $[\gamma, v]$ be the equivalence class $\Gamma_x(\gamma, v)$. The quotient $(\Gamma \times S_x)/\Gamma_x$ of this free action, denoted by $\Gamma \times_{\Gamma_x} S_x$, is a Γ-bundle with fiber S_x over $\Gamma x \cong \Gamma/\Gamma_x$. Γ acts on $\Gamma \times_{\Gamma_x} S_x$ by $(\delta, [\gamma, v]) \mapsto [\delta\gamma, v]$.

Proposition 14 (Differential slice theorem). *If* ε *is small enough, then*

$$\lambda_x : \begin{cases} \Gamma \times_{\Gamma_x} S_x \longrightarrow U_x \\ \quad [\gamma, v] \quad \longmapsto \gamma v \end{cases}$$

is a Γ*-equivariant diffeomorphism.*

Proof. See [20,5], and for the case of a reductive group [39]. □

We call a bundle $S \to \Gamma x$ satisfying the slice theorem a *slice bundle*. S is accompanied by a *tubular neighborhood* $U_x = \lambda_x(S)$, also called a *tube*. There are some characterizing properties of a slice bundle: If the intersection of $\lambda_x(S_y)$ and $\lambda_x(S_{\gamma y}) = \lambda_x(\gamma S_y) = \gamma\lambda_x(S_y)$ is non-empty, then $\gamma \in \Gamma_y$. In this case both sets coincide. Of course, the converse holds true, too. Furthermore, $\lambda_x(S_y)$ is closed in $U_x = \Gamma\lambda_x(S_y)$.

¿From the slice theorem follows that the orbit space is locally diffeomorphic to the orbit space of Γ_x, meaning that λ_x induces a homeomorphism

$$\lambda_x^* : \begin{cases} C^\infty(U_x)^\Gamma \longrightarrow C^\infty(S_x)^{\Gamma_x} \\ \quad f \qquad \longmapsto \quad f \circ \lambda_x \end{cases} .$$

As the following corollary implies, this is also valid for the embedded orbit spaces.

Corollary 15. *Let $\rho_1, \dots, \rho_k$ be a Hilbert basis of V, and $\sigma_1, \dots, \sigma_l$ a minimal Hilbert basis for the action of Γ_x on N_x. Denote the local orbit space $\sigma(N_x) \subset \mathbb{R}^l$ by Y_x. Then a C^∞-map $H_x : \mathbb{R}^l \to \mathbb{R}^k$ exists with*

1. *$H_x \circ \sigma = \rho \circ \lambda_x$ on S_x.*
2. *There are open sets $U \subset \mathbb{R}^l$ and $W \subset \mathbb{R}^k$ such that $H_x(Y_x \cap U) = H_x(\sigma(S_x)) = \rho(U_x) = Y \cap W$.*
3. *The differential $DH_x(0)$ is one-to-one.*
4. *There are open sets $U \subset \mathbb{R}^l$ and $W \subset \mathbb{R}^k$ and a C^∞-map $\widetilde{H}_x : W \to \mathbb{R}^l$ such that $\widetilde{H}_x \circ H_x|_U = \mathrm{id}_U$.*

Obviously, $l \leq r$ and $\max_i \deg \sigma_i \leq \max_i \deg \rho_i$.

We will prove this together with the next lemma.

At first, we introduce an appropriate notion of tangent spaces of the orbit space Y. Since it is possible to add relations to Q without destroying (2), P determines Q uniquely on Y, only. If the relations possess nontrivial linear parts at $y \in Y$, then $DQ(y)$ is not uniquely determined by P and (2). However, this is true for the restriction of $DQ(y)$ to the tangent space of Y defined as follows.

As above, denote the ideal of relations by I, and the variety of I by Z. The tangent space of Z at $z \in Z$ is given in terms of I in the following way: Let $r_1, \dots, r_j$ be a set of generators of I. The differentials $Dr_1(z), \dots, Dr_j(z)$ at z are linear functions on $T_z\mathbb{R}^k$, and define a subspace T_zZ called the *(Zariski) tangent space of Z at z*. For $y \in Y$ we define the *tangent space of Y at y* as $T_yY := T_yZ$. This definition is independent of the choice of the generators of I (cf. [19]).

If $f \in C^{qn}(V)^\Gamma$ and $g \in C^n(\mathbb{R}^k)$ with $g \circ \rho = f$, then $Dg(y)|_{T_yY}$ is uniquely determined by f. Also $DQ(y)|_{T_yY}$ is uniquely determined by P and (2), and respects the tangent spaces: $DQ(y)T_yY \subset T_{Q(y)}Y$.

If the Hilbert basis is minimal, then no relation possesses a nontrivial linear part at 0, and therefore, the tangent space of Y at $\rho(0)$ has dimension k (cf. Lemma 5).

There is a supplement to Corollary 15:

Lemma 16. *For the map H_x in Corollary 15 we have* $\operatorname{range} DH_x(0) = T_yY$, *where $y := \rho(x)$.*

Proof. We prove Corollary 15 and Lemma 16 together. Since $\rho \circ \lambda_x : N_x \to \mathbb{R}^k$ is a Γ_x-invariant C^∞-map, there is a C^∞-map $H_x : \mathbb{R}^l \to \mathbb{R}^k$ with $\rho \circ \lambda_x = H_x \circ \sigma$. Since ρ and σ are proper, the second claim of the Corollary follows.

Furthermore, there is a map $\tilde{\sigma} \in C^\infty(U_x, \mathbb{R}^l)^\Gamma$ with $\tilde{\sigma} \circ \lambda_x = \sigma$. According to Proposition 2 we can find a C^∞-map Λ with $\tilde{\sigma} = \Lambda \circ \rho$. Altogether, it gives

$$\sigma = \Lambda \circ \rho \circ \lambda_x = \Lambda \circ H_x \circ \sigma \ .$$

Since σ is minimal, $\Lambda\circ\rho\circ\lambda_x$ determines the linear part of $\Lambda\circ H_x$ at 0 uniquely. Thus,

$$\mathrm{id}_{\mathbb{R}^l} = \mathrm{id}_{T_0\sigma(N_x)} = D(\Lambda\circ H_x)(0) = D\Lambda(\rho(x))DH_x(0) \ .$$

Therefore, $DH_x(0)$ is one-to-one. This proves claims 3 and 4 of the corollary.

The same arguments work for Λ, and yield $DH_x(0)D\Lambda(y)|_{T_yY} = \mathrm{id}_{T_yY}$. Therefore, $D\Lambda(y) : T_yY \to T_0Y_x \cong \mathbb{R}^l$ is one-to-one.

Let r be a relation of ρ, then $r\circ H_x$ is a relation of σ, and has a vanishing linear part at $0\in\mathbb{R}^l$. Hence, $\mathrm{range}\, DH_x(0) \subset \ker Dr(H_x(0))$. Since this holds true for all relations r of ρ, $DH_x(0)$ maps into $T_yY = \bigcap_r Dr(y)$. We have the sequence of maps

$$T_0Y_x \xrightarrow{DH_x(0)} T_yY \xrightarrow{D\Lambda(y)} T_0Y_x$$

which are one-to-one. ¿From this the assertion of the lemma follows. □

By Remark 1, the dimension of T_0Y_x is independent of the choice of σ. Since the map H_x establishes a local diffeomorphism between the orbit space Y near y and the local orbit space Y_x, the dimension of T_yY is independent of the choice of σ and ρ.

Lemma 17. *Define $R_y := DH_x(0)R_0$, where $y := \rho(x)$ and $R_0 = R$ as in Remark 9. Let Q be the reduced dynamics of $P\in C^q_\Gamma(V,V)$. Then $DQ(y)$ maps R_y into $R_{Q(y)}$.*

Proof. This follows from Remark 9 and Corollary 15. □

For later use we mention another definition of the tangent space T_yY of the orbit space. Denote the maximal ideal of Γ-invariant polynomials which vanish at $x\in V$ by J_x. The tangent space of the orbit space V/Γ at Γx is the dual space of J_x/J_x^2 ([27, p. 147]). Let $\rho_1,\ldots,\rho_k$ be a minimal Hilbert basis. It generates a complement of J_0^2 in J_0. Let $p = \mathcal{J}(p)\circ\rho$ be a Γ-invariant polynomial. Due to the minimality of ρ, the linear part of $\mathcal{J}(p)$ is unique. We identify the unit vector $e_i\in T_0Y$ with the linear function which sends a polynomial p to $\partial_{\rho_i}\mathcal{J}(P)(0)$, the coefficient of the ρ_i-term in $\mathcal{J}(p)$. This function is well-defined on J_0/J_0^2, too. This identification depends on the choice of the Hilbert basis. The slice theorem allows to define such a duality also for $x\neq 0$.

Remark 18. Let K_i be the subset of J_x of homogeneous polynomials of degree i, then $(K_1\oplus K_2)/J_x^2$ is dual to $R_{\rho(x)}$.

Remark 19. Using the alternative definition of T_yY, one can view the differential $DQ(y)|_{T_yY}$, $y := \rho(x)$, of the reduced map $Q = \mathcal{J}_{\mathsf{m}}(P)$ as the dual map of

$$P^* : \begin{cases} J_{P(x)}/J^2_{P(x)} \longrightarrow J_x/J_x^2 \\ \qquad p \qquad\ \longmapsto p\circ P \end{cases} .$$

Suppose $P(0) = 0$ and that $V_1 \oplus V_2$ is a Γ-invariant decomposition of V, and V_1 is invariant with respect to $DP(0)$, where 0 is an equilibrium of P. Denote by $L_{2,0}$, $L_{1,1}$, and $L_{0,2}$ the sets of Γ-invariant, multihomogeneous, quadratic polynomials. Then $L_{0,2}$ and $L_{1,1} \oplus L_{0,2}$ are $DP(0)$-invariant, and $(L_{2,0}/J_0^2)^*$ and $((L_{2,0} \oplus L_{1,1})/J_0^2)^*$ are invariant with respect to $S := D\mathcal{J}_{\mathsf{m}}(DP(0))$. These facts can be adopted for a relative equilibrium of P using the map H_x .

Stratification by orbit type is another useful concept to describe the structure of the orbit space. The origin of a representation has the full group as isotropy group. This is, of course, also valid for the fixed points. All other points have smaller isotropy groups. The slice theorem shows that the isotropy group Γ_x cannot increase suddenly, if the point x varies. Modulo conjugation, the isotropy groups of points near an orbit Γx are subgroups of Γ_x.

The conjugation class of the isotropy subgroup is constant on an orbit ($\Gamma_{\gamma x} = \gamma \Gamma_x \gamma^{-1}$). The map $\mathcal{C} : x \mapsto [\Gamma_x]$ is Γ-invariant, where $[\Gamma_x]$ is the conjugacy class of Γ_x in Γ, called the *orbit type of* x. The fibers of $\mathcal{C}$ are Γ-invariant, homogeneous ($\lambda \neq 0 \Rightarrow \Gamma_{\lambda x} = \Gamma_x$), and are called *strata*. Sometimes the connected components of the fibers of $\mathcal{C}$ bear this name. The stratum of x is a bundle with fiber $\Gamma/\Gamma_x \cong \Gamma x$.

The strata have some nice properties following from the slice theorem:

1. The strata are locally closed smooth manifolds.
2. The boundary of a stratum is a union of other strata.
3. Let Σ_1 and Σ_2 be two distinct strata such that $\Sigma_1 \cap \overline{\Sigma_2} \neq \emptyset$, then Σ_1 is already contained in the boundary of Σ_2,
4. and there are points $x \in \Sigma_1$ and $y \in \Sigma_2$ such that Γ_y is a proper subgroup of Γ_x.
5. Let $N_x^0 := N_x^{\Gamma_x}$ be the space of fixed points in N_x. If $S_x^0 := S_x \cap N_x^0$, then $\Gamma \times_{\Gamma_x} N_x^0$ and $\Gamma \times_{\Gamma_x} S_x^0$ are trivial bundles over the orbit. We have $\lambda_x(\Gamma \times_{\Gamma_x} S_x^0) = U_x \cap \Sigma_x$, where Σ_x is the stratum containing x.
6. There is a unique stratum which is open and dense in V, called the *principal stratum*.
7. There are at most finitely many strata in V.

$\mathcal{C}$ and the Hilbert map ρ induce a stratification of the orbit space Y. The ρ-images of the strata of V are strata of Y with 1.–3. and 6. This stratification coincides with the canonical stratification of Y as a semialgebraic set ([4]).

Denote by Σ_x the stratum containing x. The image of the map $D\rho(x) : T_xV \to T_y\mathbb{R}^k$ is the tangent space $T_y\Upsilon$ of the stratum $\Upsilon := \rho(\Sigma_x) \subset Y$ at $y := \rho(x)$. Indeed, $D\rho(x)(T_x\Sigma_x) = T_y\Upsilon$. If Σ_x is the principal stratum, then Υ is the principal stratum of Y, and the tangent spaces of Υ and Y coincide: $T_y\Upsilon = T_yY$. Otherwise $T_y\Upsilon$ is a proper subset of T_yY.

$T_y\Upsilon$ covers the directions inside the stratum Υ, while T_yY contains also the directions out of Υ but inside the orbit space.

The stratification can be useful for investigating equivariant dynamics. A Γ-equivariant vector field F is tangent to the strata, i.e., $F(x) \in T_x\Sigma_x$. The reduced vector field $G := \mathcal{J}_\rho(F)$ is also tangent to the strata of Y, because

$$G \circ \rho = D\rho(x)F(x) \tag{5}$$

and $D\rho(x)T_x\Sigma_x = T_y\Upsilon$. A result by Bierstone [4] and Schwarz [38] states that each smooth vector field G on $\mathbb{R}^k$ which is tangent to the strata of Y can be lifted to a smooth Γ-equivariant vector field F on V such that (5) holds.

Similar things hold for a Γ-equivariant map P. The isotropy group of the image $P(x)$ of a point x contains Γ_x, because of $\gamma P(x) = P(\gamma x) = P(x)$ for $\gamma \in \Gamma_x$. Therefore, fixed point spaces are invariant, and $P(x)$ lies in the closure of Σ_x. If P is one-to-one, then the strata are invariant.

The map H_x, defined in Corollary 15, sends the strata of the local orbit space to strata of Y, locally. As a byproduct we make the remark

Remark 20. Over each stratum of the orbit space there are two continuous bundles with fibers T_zY and R_z, respectively. They are invariant with respect to reduced diffeomorphisms.

Linearizations along an invariant orbit and eigenvalues at a relative equilibrium First we consider an invariant orbit Γx of the vector field $F \in \mathrm{Vec}^q_\Gamma(V)$, i.e., $F(x) \in T_x\Gamma x$. The slice theorem provides coordinates near Γx. Locally, the space is split into directions tangent to the orbit and normal directions. This yields a decomposition of the vector field $F = F_{\mathrm{T}} + F_{\mathrm{N}}$ in a tangent part F_{T} and a normal part F_{N} such that $F_{\mathrm{T}}(z) \in T_z\Gamma z$ and $F_{\mathrm{N}}(z) \in S_y$ ([24]), for $z \in S_y$ and $y \in \Gamma x$. Moreover, this decomposition can be achieved in such a way that F_{N} is a Γ_y-equivariant vector field on S_y.

Proposition 21. *Let Γx be an invariant orbit of $F \in \mathrm{Vec}^q_\Gamma(V)$. Then there are a slice bundle $S \to \Gamma x$ with tube $\lambda_x(S) = U_x$ and two vector fields F_{T}, $F_{\mathrm{N}} \in \mathrm{Vec}^q_\Gamma(U_x)$ with*

1. $F|_{U_x} = F_{\mathrm{T}} + F_{\mathrm{N}}$.
2. $\forall z \in U_x : F_{\mathrm{T}}(z) \in T_z\Gamma z$.
3. $\forall y \in \Gamma x, z \in S_y : F_{\mathrm{T}}(z) \in N_y$.

If $y \in \Gamma x$, then $F_{\mathrm{N}}|_{S_y} \in \mathrm{Vec}^q_{\Gamma_y}(S_y)$ has an equilibrium at $0 \in S_x$.

Proof. The proofs for C^∞-vector fields in [14,24] also work in the present case. ☐

For a given relative equilibrium x we consider the vector field $\widetilde{F} := F_{\mathrm{N}}|_{S_x} \in \mathrm{Vec}^q_{\Gamma_x}(S_x)$. Since F_{T} influences only the flow inside the orbits, it does not affect the stability of the invariant orbit Γx. But F_{N}, and therefore $\widetilde{F}$, describe whether the distance between Γx and a nearby trajectory of F increases or decreases, and thus reflect the stability of the invariant orbit (cf. the discussion for maps below). Thus, we use $\widetilde{F}$ to describe the stability of the relative equilibrium x, and regard its linearization at 0 as the linearization of F along Γx.

Note that F_{T} and F_{N} are not unique, in general. However, if G is the reduced vector field, then $G \circ \rho = D\rho F = D\rho(F_{\mathrm{N}} + F_{\mathrm{T}}) = D\rho F_{\mathrm{N}}$. Now we are ready to apply Theorem 13 to $\widetilde{F}$ (cf. [23]).

Theorem 22. *Let $F \in \mathrm{Vec}^q_\Gamma(V)$, and $G \in \mathrm{Vec}^1(\mathbb{R}^k)$ be a reduced vector field with (5). Suppose Γx is an invariant orbit of F, and, therefore, $y := \rho(x)$ is an equilibrium of G. Define $\widetilde{F}$ be as above. Then the eigenvalues of $DG(y)|_{T_yY}$ are sums of the eigenvalues of $D\widetilde{F}(0)$. Moreover, if λ is an eigenvalue of $D\widetilde{F}(0)$, then $DG(y)|_{T_yY}$ has the eigenvalue λ or $\lambda + \bar{\lambda}$.*

Remark 23. This statement remains true, if $DG(y)|_{R_y}$ replaces $DG(y)|_{T_yY}$, where R_y is as in Lemma 17.

Proof. Assume a minimal Hilbert map σ on N_x. There is a reduced vector field $G_\sigma \in \mathrm{Vec}^1(W)$ with $G_\sigma \circ \sigma = D\sigma\widetilde{F}$, where W is an open set with $W \cap \sigma(N_x) = \sigma(S_x)$. The relation between G_σ and G is given by

$$G \circ H_x \circ \sigma = G \circ \rho \circ \lambda_x = D\rho F = DH_x\, D\sigma F = (DH_x\, G_\sigma) \circ \sigma \ ,$$

where H_x is as in Corollary 15. As σ is minimal, this yields $D(G \circ H_x)(0) = D(DH_x\, G_\sigma)(0) = DH_x(0)DG_\sigma(0) + DDH[G_\sigma(0), v]$. Since the last term vanishes, we obtain $DG(y)DH_x(0) = DH_x(0)DG_\sigma(0)$.

Since $DH(0)$ is one-to-one onto T_yY, the spectra of $DG(y)|_{T_yY}$ and $DG_\sigma(0)$ coincide. Now applying Theorem 13 to G_σ proves the claims. □

Finally we carry out the analogous analysis for a Γ-equivariant map P. The splitting of maps is a little bit harder than for vector fields. We are searching for a map $\widetilde{P}$ on the slice S_x such that P is the combination of $\widetilde{P}$ and a drift along the orbits. A way to define $\widetilde{P}$ is to shift $P(y)$ along the orbit $\Gamma P(y)$. If y is close enough to the invariant orbit, then $\Gamma P(y)$ meets the slice S_x, and we may take a point of intersection for $\widetilde{P}(y)$. Unfortunately, we cannot guarantee that $\widetilde{P}$ obtained this way is Γ_x-equivariant. Therefore, we have to be more careful.

Proposition 24. *Let Γx be an invariant orbit of the map $P \in C^q_\Gamma(V,V)$. There is a smallest number a such that an element c in the centralizer $C(\Gamma_x)$ of Γ_x exists with $P^a(x) = cx$. Then a slice bundle $S \to \Gamma x$ with tube U_x exists, and there are maps $\widetilde{P} : S_x \to N_x$ and $\gamma : S_x \to \Gamma$ with*

1. *$\forall y \in S_x : P^a(y) = \gamma(y)(\widetilde{P}(y))$.*
2. *$\widetilde{P} \in C^q_{\Gamma_x}(S_x, N_x)$.*

3. $\forall y \in S_x : \gamma(y) \in C^q(\Gamma_x)$.

Proof. The proofs in [14] or [28] for the C^∞ case remain valid in the present case. □

The stability of the invariant orbit Γx with respect to P is equivalent to the stability of the equilibrium 0 with respect to $\widetilde{P}$: Given a neighborhood U of Γx, then there is a neighborhood U' of 0 in S_x such that $\Gamma(U') \subset U$. Suppose the equilibrium $0 = \widetilde{P}(0)$ is stable, i.e., a neighborhood V' exists such that $\widetilde{P}^n(y) \in U'$ for all $n \in \mathbb{N}_0$ and $y \in V'$. Then $P^{an}(\Gamma(V')) = \Gamma P^{an}(V') = \Gamma \widetilde{P}^n(V') \subset U$. Choose a neighborhood V of Γx such that $P^n(V) \subset \Gamma V'$ for all $n \leq a$, then $P^n(V) \subset U$ for all $n \in \mathbb{N}$, i.e., Γx is stable. Also the converse is true.

Hence, $\widetilde{P}$ provides an appropriate description of the stability of the relative equilibrium x. We take its linearization at 0 for the linearization of P along Γx. If x is even an equilibrium of P, then the eigenvalues of $DP(x)|_{T_x\Gamma_x}$ have modulus 1. And the a-th power of the other eigenvalues of $DP(x)$ are the eigenvalues of $D\widetilde{P}(0)$.

Theorem 25 ([32]). *Let P and Q be as in (2). Suppose Γx is an invariant orbit of P, and therefore $y := \rho(x)$ is an equilibrium of Q. Let $\widetilde{P}$ be as in Proposition 24.*
Let U be either the open or the closed unit disc or the complement of one of them. Then the eigenvalues of $DQ(y)|_{T_yY}$ lie in U, if and only if the same holds true for $D\widetilde{P}(0)$. This statement remains valid, if $DQ(y)|_{R_y}$ replaces $DQ(y)|_{T_yY}$, where R_y is taken from Lemma 17.

Proof. The line of arguments is very similar to the proof of Theorem 22. Using Theorem 6 and its Corollaries, and Corollary 15, we establish a correspondence between the eigenvalues of $D\widetilde{P}(0)$ and $DQ(y)|^a_{T_yY}$. Note that the eigenvalues of $DQ(y)^a$ are a-th powers of the eigenvalues of $DQ(y)$. □

This theorem provides a criterion for asymptotic stability and instability of a relative equilibrium in terms of the reduced map. We point out that $\widetilde{P}$ is used only to prove, but not to apply the criterion. It can also be used to detect, when an asymptotically stable invariant orbit becomes unstable. We refer also to the Section 8 and the discussion after Corollary 8.

5 Relative periodic points and Poincaré maps

In this section we use the last theorem to discuss relative periodic points. We consider a vector field $F \in \mathrm{Vec}^q_\Gamma(V)$ and its reduction to the orbit space $G := \mathcal{J}_\rho(F)$. Let $y \in Y$ be a periodic point of G, i.e., the trajectory through y is closed. The preimage Ω of its trajectory is a compact manifold in V. It is invariant with respect to both the flow Φ_t induced by F and the action of

Γ. The points of Ω are called *relative periodic points.* Ω is contained in some stratum.

To investigate the stability of a periodic trajectory, one usually constructs a Poincaré map. This is the first return map Q on a hypersurface E transversal to the trajectory. The point y is an equilibrium of Q on E, whose stability reflects the stability of the periodic trajectory.

For the manifold Ω we construct a similar first return map (see [13]), and relate it to Q. The orthogonal complement $M\Omega$ of the tangent bundle $T\Omega$ in $TV|_\Omega$ has the fibers $M_z = T_z\Omega^\perp$. $M\Omega$ is Γ-invariant.

Take $x \in \Omega$. Let $S \to \Gamma x$ be a slice bundle with the tubular neighborhood U_x. The subbundle $R \to \Gamma x$ with fibers $R_z := S_z \cap M_z$ gives a Γ-invariant manifold $D := \lambda_x(R)$, where λ_x is as in the slice theorem.

Lemma 26. *If the slices are small enough, D is a Γ-invariant hypersurface transversal to F.*

Proof. Take the orthogonal projection v of $F(x)$ into $N_x = (T_x\Gamma x)^\perp$. Since M_x is a codimension one subspace of N_x, and is orthogonal to $F(x)$, v is also orthogonal to M_x, and is not the zero vector. As $F(x)$ is Γ_x-invariant, the vector v and, therefore, the function $\varphi(w) := \langle w, v\rangle$ are also Γ_x-invariant. Using λ^*, φ can be extended to a Γ-invariant function near Γx. Since, for $w \in N_x$, $\varphi(w) = 0$ is equivalent to $w \in M_x$, the Γ-invariance of φ yields $\forall z \in U_x : \varphi(z) = 0 \Leftrightarrow z \in D$. Furthermore, $\langle \operatorname{grad}\varphi(x), F(x)\rangle = \langle v, F(x)\rangle \neq 0$. □

D intersects Ω along the orbit Γx transversally. Furthermore, $T_zD = M_z \oplus T_z\Gamma z$.

Let θ be the period of y. Then $\Phi_\theta(x)$ meets the orbit at some γx. Since $\varphi(\gamma x) = 0$ and

$$\frac{d}{dt}\varphi(\Phi_\theta(x)) = \langle \operatorname{grad}\varphi(\gamma x), \dot{\Phi}_\theta(x)\rangle = \langle (T_x\gamma)v, (T_x\gamma)F(x)\rangle = \langle v, F(x)\rangle \neq 0\ ,$$

applying the implicit function theorem to $(t, z) \mapsto \varphi(\Phi_t(z))$ gives a Γ-invariant function τ in a neighborhood of Γx, so that $\varphi(\Phi_{\tau(z)}(z)) = 0$.

If D' is the intersection of the domain of τ with D, then define the map

$$P : \begin{cases} D' \longrightarrow & D \\ z \longmapsto & \Phi_{\tau(z)}(z) \end{cases} .$$

P is Γ-equivariant and as smooth as F. Furthermore, Γx is an invariant orbit of P. We call (D, D', P, τ) *Poincaré system.*

The next proposition establishes a correspondence between a Poincaré map Q to the periodic point y of G and P.

Proposition 27. *Assume x to be a relative periodic point of $F \in \mathrm{Vec}^q_\Gamma(V)$ with Poincaré system (D, D', P, τ). At the periodic point $y := \rho(x)$ of the reduced system G, there is a transversal section E and a Poincaré map Q with $\rho^{-1}(E) \subset D$ and $\rho \circ P = Q \circ \rho$.*

Proof. By Proposition 2, a C^∞-function ψ exists in a neighborhood of y with $\psi \circ \rho = \varphi$. Define $E := \{z \in \mathbb{R}^k \mid \psi(z) = 0 \wedge \operatorname{grad}\psi(z) \neq 0\}$. Since

$$D\psi(y)G(y) = D\psi(y)D\rho(x)F(x) = D(\psi \circ \rho)(x)F(x) = D\varphi(x)F(x) \neq 0 ,$$

$\operatorname{grad}\psi(y)$ is not orthogonal to $G(y)$, and E is a section transversal to the trajectory of y.

Let $z \in V$ with $\rho(z) \in E$, then $\psi(\rho(z)) = \varphi(z) = 0$ and, therefore, $z \in D$. So $\rho^{-1}(E) \subset D$.

The flow of G is denoted by Ψ_t, and has the property $\Psi_t \circ \rho = \rho \circ \Phi_t$. Let Q be the Poincaré map on E. The trajectory of $\rho(z) \in E$, $z \in D$, meets E at the time $\tau(z)$, because $\rho(\Phi_{\tau(z)}(z)) \in \rho(D) \subset E$, but not before $\tau(z)$. So $Q(\rho(z)) = \rho(\Phi_{\tau(z)}(z)) = \rho(P(z))$. □

Theorem 25 applies to P. In particular, we have

Theorem 28. *In the situation of the last proposition, set $T := T_yY \cap T_yE$. For $\widetilde{P}$ as in Proposition 24, the eigenvalues of $D\widetilde{P}(0)$ are contained in $\{z \mid |z| < 1\}$, ($\{z \mid |z| \leq 1\}$, $\{z \mid |z| > 1\}$, or $\{z \mid |z| \geq 1\}$), if and only if the eigenvalues of $DQ(0)|_T$ do so.*

Using this theorem, the stability of relative periodic points is determinable in terms of eigenvalues of the linearization of any Poincaré map on the orbit space. Recall that Poincaré maps are related by conjugacy.

6 Lyapunov exponents

Lyapunov exponents are a generalization of eigenvalues at equilibria. One averages the separation of neighboring trajectories during time evolution. The Lyapunov exponents are given by the average rate of expansion of distance vectors $v \neq 0$, e.g.

$$\lambda_v(x) := \lim_{n\to\infty} \frac{1}{n} \log \|DP^n(x)v\|$$

is a Lyapunov exponent of a map P at x. By Oseledec's multiplicative ergodic theorem, this limit exists almost everywhere with respect to an invariant measure ([3]), and at most $\dim V$ many different values occur, if the measure is ergodic.

Concerning orbit space reduction, for Lyapunov exponents the same results hold as presented above for eigenvalues of vector fields. In [31] it is proved that the Lyapunov exponents of a reduced map Q are $\mathbb{N}_0$-linear combinations of the exponents of the original map P. Moreover, if λ is an exponent of P, then λ or 2λ is an exponent of Q. Again, only those Lyapunov exponents of the reduced system are relevant which belong to the tangent space of the orbit space. By the way, here the invariant measure on the orbit is supposed to be the measure induced by the invariant measure of the original system via the Hilbert map.

Of course, eigenvalues and Lyapunov exponents have a lot of common aspects. However, the case of Lyapunov exponents is much more involved. Concerning eigenvalues, our argument in Sect. 3 is more or less elementary due to the fact that the general eigenvalue case can be reduced to the case, where the differential of the reduced map is diagonal and, therefore, self adjoint. It is not obvious how to achieve such a diagonalization in the general case of Lyapunov exponents. However, using the duality principle indicated in Remark 19 and the bundle picture as outlined throughout the paper, the argument in Sect. 3 can be carried over to the case of Lyapunov exponents. In particular, estimates are needed in order to exclude the influence of nonlinear terms in the original system, i.e., for the block matrix in (3) the asymptotic growth of the blocks above the diagonal as $t \to \infty$ must be estimated along the basic trajectory. They must not grow exponentially fast with respect to some norm.

7 Example

Let $P := \{\cos(kx), \sin(kx) \mid k \in \mathbb{N}_0\}$, and denote the span of P in the Sobolev space $H^m(]0, 2\pi[)$ by H^m_{per}. Set $Z := (H^0_{\mathrm{per}})^4$, and let $D(C) := (H^4_{\mathrm{per}})^4$ be the domain of definition of the closed operator C in Z given by the linearization at $u = 0$ of the right-hand side of the partial differential equation

$$u_t = -\partial_x^4 u - D\Delta u + \Lambda u + f(u, u_x) \ . \tag{6}$$

Here $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^4$ is supposed to fulfill periodic boundary conditions $u(t, x) = u(t, x + 2\pi)$, and $f(u, u_x) = o(|u| + |u_x|)$ to be sufficiently smooth. Furthermore, we assume that (6) is invariant with respect to the flip $x \mapsto -x$ and the following action of the torus group T^2:

$$\begin{cases} T^2 \times \mathbb{R}^4 & \longrightarrow \qquad\qquad \mathbb{R}^4 \\ ((\vartheta_1, \vartheta_2), u) & \longmapsto (R(-\vartheta_1)(u_1, u_2)^t, R(-\vartheta_2)(u_3, u_4)^t) \end{cases}$$

Therefore, restrictions for D, Λ, and f arise. We consider the constant, real matrices

$$\Lambda = \begin{pmatrix} \lambda & -\omega_1 & 0 & 0 \\ \omega_1 & \lambda & 0 & 0 \\ 0 & 0 & \sigma & -\omega_2 \\ 0 & 0 & \omega_2 & \sigma \end{pmatrix}$$

and $D = \mathrm{diag}(d_1, d_1, d_2, d_2)$. The T^2-invariants and T^2-equivariants are

$$\begin{aligned} e_1 &= u_1^2 + u_2^2 & \qquad e_3 &= u_3^2 + u_4^2 \\ e_2 &= {u_x}_1^2 + {u_x}_2^2 & \qquad e_4 &= {u_x}_3^2 + {u_x}_4^2 \\[1ex] e_5 &= u_1 u_{x1} + u_2 u_{x2} & \qquad e_7 &= u_3 u_{x3} + u_4 u_{x4} \\ e_6 &= -u_1 u_{x2} + u_2 u_{x1} & \qquad e_8 &= -u_3 u_{x4} + u_4 u_{x3} \end{aligned}$$

$$E_1 = \begin{pmatrix} u_1 \\ u_2 \\ 0 \\ 0 \end{pmatrix} \qquad E_2 = \begin{pmatrix} -u_2 \\ u_1 \\ 0 \\ 0 \end{pmatrix} \qquad E_3 = \begin{pmatrix} 0 \\ 0 \\ u_3 \\ u_4 \end{pmatrix} \qquad E_4 = \begin{pmatrix} 0 \\ 0 \\ -u_4 \\ u_3 \end{pmatrix}$$
$$E_5 = \begin{pmatrix} u_{x1} \\ u_{x2} \\ 0 \\ 0 \end{pmatrix} \qquad E_6 = \begin{pmatrix} -u_{x2} \\ u_{x1} \\ 0 \\ 0 \end{pmatrix} \qquad E_7 = \begin{pmatrix} 0 \\ 0 \\ u_{x3} \\ u_{x4} \end{pmatrix} \qquad E_8 = \begin{pmatrix} 0 \\ 0 \\ -u_{x4} \\ u_{x3} \end{pmatrix}$$

Changing the sign of x keeps $e_1, \ldots, e_4$ and $E_1, \ldots, E_4$ invariant, and switches the sign of $e_5, \ldots, e_8$ and $E_5, \ldots, E_8$. Therefore, the general form of the nonlinearity f is $\sum f_i(e_1, \ldots, e_8)E_i$, where $e_5, \ldots, e_8$ and $E_5, \ldots, E_8$ appear only in pairs.

In particular, $u = 0$ is always an equilibrium of (6). The eigenvalues of C are $\alpha^k_\pm = \lambda - k^4 + d_1 k^2 \pm \mathrm{i}\omega_1$ and $\beta^k_\pm = \sigma - k^4 + d_2 k^2 \pm \mathrm{i}\omega_2$, $k \in \mathbb{N}_0$, with eigenvectors

$$w_1^k = \begin{pmatrix} \mathrm{i}\mathrm{e}^{\mathrm{i}kx} \\ \mathrm{e}^{\mathrm{i}kx} \\ 0 \\ 0 \end{pmatrix} \quad w_2^k = \begin{pmatrix} \mathrm{i}\mathrm{e}^{-\mathrm{i}kx} \\ \mathrm{e}^{-\mathrm{i}kx} \\ 0 \\ 0 \end{pmatrix} \quad w_3^k = \begin{pmatrix} -\mathrm{i}\mathrm{e}^{\mathrm{i}kx} \\ \mathrm{e}^{\mathrm{i}kx} \\ 0 \\ 0 \end{pmatrix} \quad w_4^k = \begin{pmatrix} -\mathrm{i}\mathrm{e}^{-\mathrm{i}kx} \\ \mathrm{e}^{-\mathrm{i}kx} \\ 0 \\ 0 \end{pmatrix}$$
$$w_5^k = \begin{pmatrix} 0 \\ 0 \\ \mathrm{i}\mathrm{e}^{\mathrm{i}kx} \\ \mathrm{e}^{\mathrm{i}kx} \end{pmatrix} \quad w_6^k = \begin{pmatrix} 0 \\ 0 \\ \mathrm{i}\mathrm{e}^{-\mathrm{i}kx} \\ \mathrm{e}^{-\mathrm{i}kx} \end{pmatrix} \quad w_7^k = \begin{pmatrix} 0 \\ 0 \\ -\mathrm{i}\mathrm{e}^{\mathrm{i}kx} \\ \mathrm{e}^{\mathrm{i}kx} \end{pmatrix} \quad w_8^k = \begin{pmatrix} 0 \\ 0 \\ -\mathrm{i}\mathrm{e}^{-\mathrm{i}kx} \\ \mathrm{e}^{-\mathrm{i}kx} \end{pmatrix},$$

where w_1^k and w_2^k belongs to α^k_+, and so on. The eigenvalues are double for $k \in \mathbb{N}$, and simple for $k = 0$. We introduce $v_1^k = (w_1^k + w_4^k)/2$, $v_2^k = (w_1^k - w_4^k)/2\mathrm{i}$, $v_3^k = (w_2^k + w_3^k)/2$, $v_4^k = (w_2^k - w_3^k)/2\mathrm{i}$, $v_5^k = (w_5^k + w_8^k)/2$, $v_6^k = (w_5^k - w_7^k)/2\mathrm{i}$, $v_7^k = (w_6^k + w_7^k)/2$, and $v_8^k = (w_6^k - w_8^k)/2\mathrm{i}$. These vectors form a Hilbert space basis of both $D(C)$ and Z.

We have the following action of $\Gamma := \mathrm{O}(2) \times T^2$ on Z: $\varphi \in \mathrm{SO}(2)$ acts on u by $\varphi \cdot u : (t, x) \mapsto u(t, x - \varphi)$. Fix $\kappa \in \mathrm{O}(2) \setminus \mathrm{SO}(2)$, and let its action be $\kappa u : (t, x) \mapsto u(t, 2\pi - x)$. For $k \in \mathbb{N}$, set $v^k = (v_1^k, v_2^k, v_3^k, v_4^k, v_5^k, v_6^k, v_7^k, v_8^k)^t$, then

$$\varphi v^k = \begin{pmatrix} R(-k\varphi) & & & \\ & R(k\varphi) & & \\ & & R(-k\varphi) & \\ & & & R(k\varphi) \end{pmatrix} v^k \qquad \text{and} \qquad \kappa v^k = \begin{pmatrix} & I & & \\ I & & & \\ & & & I \\ & & I & \end{pmatrix} v^k ,$$

with $R(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$ and $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and

$$(\vartheta_1, \vartheta_2) v^k = \begin{pmatrix} R(-\vartheta_1) & & & \\ & R(-\vartheta_1) & & \\ & & R(-\vartheta_2) & \\ & & & R(-\vartheta_2) \end{pmatrix} v^k ,$$

and similarly for $k = 0$.

Up to degree 5, f has the general form

$$
\begin{aligned}
f(u) \;=\; & \sum_{i,j=1}^{4} f_{ij} e_j(u) E_i(u) + \sum_{i,j=5}^{8} f_{ij} e_j(u) E_i(u) \\
& + \sum_{j,k=1}^{4} \sum_{i=1}^{j} f_{kij} e_i(u) e_j(u) E_k(u) + \sum_{k=1}^{4} \sum_{j=5}^{8} \sum_{i=5}^{j} f_{kij} e_i(u) e_j(u) E_k(u) \\
& + \sum_{j,k=5}^{8} \sum_{i=1}^{4} f_{kij} e_i(u) e_j(u) E_k(u) \; .
\end{aligned}
$$

We write

$$
u = \sum_{i=1}^{4} b_i^{(0)} v_i^0 + \sum_{k=1}^{\infty} \sum_{i=1}^{8} b_i^{(k)} v_i^k \; .
$$

The action of Γ on the coordinates is the transpose of the action on the basis vectors v.

Equations of type (6) appear as mathematical models in various contexts of pattern formation (see e.g. [41, Ch. III. 4]). Here the SO(2) component of the symmetry group Γ is related to the periodic boundary condition imposed above, while the remaining symmetries are either of geometric nature or normal form symmetries. In the following we are going to study symmetry breaking bifurcations from the basic equilibrium $u = 0$ of this problem. Of course, $u = 0$ has the symmetry of the full group Γ. We use λ and σ as bifurcation parameters, while the remaining parameters in the problem are supposed to be fixed. A few restrictions for the latter will be stated where needed.

We do not intend to analyze the bifurcation behavior of this problem as complete as possible here. In order to illustrate our general theory, we rather restrict ourselves to a specific bifurcation scenario including many aspects of the theory developed before. To this end we consider a critical point ($\lambda_0 = 1 - d_1$, $\sigma_0 = 1 - d_2$) in the parameter plane, where the basic equilibrium $u = 0$ becomes unstable simultaneously with respect to all modes v_1^1, …, v_8^1 with wave number $k = 1$, while it remains stable with respect to all other modes v_i^k ($1 < d_1, d_2 < 5$). Also we assume $\omega_1 \omega_2 \neq 0$, and $\omega_1 \neq \omega_2$.

Near such a critical point in the parameter space and for u near 0 the original problem can be reduced to a Γ-equivariant first order system of ordinary differential equations on an 8-dimensional local center manifold.

The map h which describes the center manifold as a graph in Z over the center space $\mathrm{span}\{v_1^1, \ldots, v_8^1\}$ of C is Γ-equivariant. Due to the T^2-action, h has only terms of odd degree. Therefore, the reduced system up to degree 3 can be computed without knowledge of h. Since the terms of f are only of odd degree, the next non-vanishing terms of the reduced system have degree 5. To compute them the cubic terms of h are needed. The action of SO(2) implies that the image of h is contained in the space generated by $\{v_i^k \mid k \text{ odd}, k > 1\}$. Moreover, the image of the cubic terms is contained in $\mathrm{span}\{v_1^3, \ldots, v_8^3\}$.

The coordinates of the center space are $b_1^{(1)}, \ldots, b_8^{(1)}$, $\lambda' = \lambda + d_1 - 1$, $\sigma' = \sigma + d_2 - 1$. The vector field on the center manifold is Γ-equivariant. Therefore, we can perform a further reduction by the orbit space method. For the sake of simplicity, we use complex notation $z_1 = b_1^{(1)} + \mathrm{i} b_1^{(1)}$, $z_2 = b_3^{(1)} + \mathrm{i} b_4^{(1)}$, and so on. A minimal Hilbert basis is given by

$$\begin{aligned}
\rho_1 &= z_1\bar{z}_1 + z_2\bar{z}_2 & \rho_5 &= \delta_1\delta_2 \\
\rho_2 &= z_3\bar{z}_3 + z_4\bar{z}_4 & \rho_6 &= \tfrac{1}{2}(z_1\bar{z}_2\bar{z}_3 z_4 + \bar{z}_1 z_2 z_3 \bar{z}_4) \\
\rho_3 &= \delta_1^2 & \rho_7 &= \tfrac{1}{2}\delta_1(z_1\bar{z}_2\bar{z}_3 z_4 - \bar{z}_1 z_2 z_3 \bar{z}_4) \\
\rho_4 &= \delta_2^2 & \rho_8 &= \tfrac{1}{2}\delta_2(z_1\bar{z}_2\bar{z}_3 z_4 - \bar{z}_1 z_2 z_3 \bar{z}_4)
\end{aligned}$$

where $\delta_1 := z_1\bar{z}_1 - z_2\bar{z}_2$ and $\delta_2 := z_3\bar{z}_3 - z_4\bar{z}_4$. This Hilbert basis has several relations. The orbit space Y is five dimensional and has the following strata (The hierarchy of the strata reflects the fact that the components (u_1, u_2) and (u_3, u_4) are interchangeable in (6) from the qualitative point of view.):

Stratum	Isotropy*	$\dim \Upsilon_i$ $\dim T_y Y$	in the closure of**
$\Upsilon_0 := \{0\}$	$\mathrm{O}(2) \times T^2$	0 8	$\Upsilon_1, \Upsilon_2, \Upsilon_4, \Upsilon_5$
$\Upsilon_1 := \{\rho_1 > 0, \rho_2 = \cdots \rho_8 = 0\}$	$\vartheta_2 \in S^1, \kappa$	1 8	$\Upsilon_3, \Upsilon_7, \Upsilon_8$
$\Upsilon_2 := \{\rho_1 > 0, \rho_3 = \rho_1^2, \rho_i = 0\}$	$\vartheta_1 = -\varphi, \vartheta_2 \in S^1$	1 6	$\Upsilon_3, \Upsilon_9, \Upsilon_{10}$
$\Upsilon_3 := \{\rho_1 > 0, \rho_1^2 > \rho_3 > 0, \rho_i = 0\}$	$\vartheta_2 \in S^1$	2 6	Υ_{11}
$\Upsilon_4 := \{\rho_2 > 0, \rho_i = 0\}$	$\vartheta_1 \in S^1, \kappa$	1 8	$\Upsilon_6, \Upsilon_7, \Upsilon_8$
$\Upsilon_5 := \{\rho_2 > 0, \rho_4 = \rho_2^2, \rho_i = 0\}$	$\vartheta_1, \vartheta_2 = -\varphi \in S^1$	1 6	$\Upsilon_6, \Upsilon_9, \Upsilon_{10}$
$\Upsilon_6 := \{\rho_2 > 0, \rho_2^2 > \rho_4 > 0, \rho_i = 0\}$	$\vartheta_1 \in S^1$	2 6	Υ_{11}
$\Upsilon_7 := \{\rho_1, \rho_2 > 0, 4\rho_6 = \rho_1\rho_2, \rho_i = 0\}$	κ	2 8	Υ_{11}
$\Upsilon_8 := \{\rho_1, \rho_2 > 0, 4\rho_6 = -\rho_1\rho_2, \rho_i = 0\}$	$\kappa(0,0,\pi)$	2 8	Υ_{11}
$\Upsilon_9 := \{\rho_1, \rho_2 > 0, \rho_3 = \rho_1^2, \rho_4 = \rho_2^2, \rho_5 = \rho_1\rho_2, \rho_i = 0\}$	$\vartheta_1 = \vartheta_2 = -\varphi \in S^1$	2 6	Υ_{11}
$\Upsilon_{10} := \{\rho_1, \rho_2 > 0, \rho_3 = \rho_1^2, \rho_4 = \rho_2^2, \rho_5 = -\rho_1\rho_2, \rho_i = 0\}$	$\vartheta_1 = -\vartheta_2 = -\varphi \in S^1$	2 6	Υ_{11}
$\Upsilon_{11} :=$ rest	*	5 5	

* This column contains generators of one representative of the conjugacy class of the stratum. κ is a fixed element of $\mathrm{O}(2) \setminus \mathrm{SO}(2)$, φ parametrizes $\mathrm{SO}(2)$, and $(\vartheta_1, \vartheta_2)$, $\vartheta_i \in S^1$, parametrizes T^2. Since all isotropy subgroups contain the element $\varphi = \vartheta_1 = \vartheta_2 = \pi$, this is omitted in the table.

** If a stratum lies in the closure of another one, then it is contained in the closure of any stratum which contains the latter.

We computed the reduced vector field using the computer algebra system Singular [18]. The part which arises from the linear and cubic terms of the right hand side of (6) is the following:

$$\left(\begin{array}{l}
2f_{68}y_5 + (4f_{13} + 4f_{57} - 4f_{14})y_6 + (f_{12} + 3f_{11} + f_{55})y_1^2 + (2f_{14} + 2f_{13})y_2y_1 \\
\qquad + (-f_{11} + 2f_{66} - f_{55} + f_{12})y_3 + 2y_1(\lambda - d_1) \\
2y_5f_{86} + (-4f_{32} + 4f_{31} + 4f_{75})y_6 + (2f_{31} + 2f_{32})y_2y_1 + (3f_{33} + f_{77} + f_{34})y_2^2 \\
\qquad + 2y_2(\sigma - d_2) + (-f_{33} + 2f_{88} - f_{77} + f_{34})y_4 \\
(-8f_{23} - 8f_{67} + 8f_{24})y_7 + (4f_{14} + 4f_{13})y_3y_2 + 4y_1y_5f_{68} \\
\qquad + (4f_{11} + 4f_{66} + 4f_{12})y_3y_1 + 4y_3(\lambda - d_1) \\
(8f_{41} + 8f_{85} - 8f_{42})y_8 + 4y_2y_5f_{86} + (4f_{32} + 4f_{31})y_4y_1 \\
\qquad + (4f_{33} + 4f_{88} + 4f_{34})y_4y_2 + 4y_4(\sigma - d_2) \\
(2(\sigma - d_2) + 2(\lambda - d_1))y_5 + (4f_{41} - 4f_{42} + 4f_{85})y_7 + (4f_{24} - 4f_{23} - 4f_{67})y_8 \\
\qquad + 2y_2y_3f_{86} + (2f_{11} + 2f_{66} + 2f_{31} + 2f_{12} + 2f_{32})y_5y_1 \\
\qquad + (2f_{34} + 2f_{88} + 2f_{14} + 2f_{13} + 2f_{33})y_5y_2 + 2y_1y_4f_{68} \\
(2(\sigma - d_2) + 2(\lambda - d_1))y_6 + (f_{22} - f_{65} - f_{21} + 2f_{76} - 2f_{56})y_7 \\
\qquad + (2f_{78} - 2f_{58} + f_{87} - f_{44} + f_{43})y_8 \\
\qquad + (-\frac{1}{4}f_{75} - \frac{1}{4}f_{31} + \frac{1}{4}f_{32})y_3y_2 + (-\frac{1}{4}f_{13} + \frac{1}{4}f_{14} - \frac{1}{4}f_{57})y_4y_1 \\
\qquad + (2f_{31} + 3f_{11} + f_{12} + f_{55} + 2f_{32})y_6y_1 + (\frac{1}{4}f_{57} + \frac{1}{4}f_{13} - \frac{1}{4}f_{14})y_2^2y_1 \\
\qquad + (f_{77} + f_{34} + 2f_{13} + 2f_{14} + 3f_{33})y_6y_2 + (\frac{1}{4}f_{31} - \frac{1}{4}f_{32} + \frac{1}{4}f_{75})y_2y_1^2 \\
(2(\sigma - d_2) + 4(\lambda - d_1))y_7 + (4f_{23} + 4f_{67} - 4f_{24})y_6^2 + (\frac{1}{2}f_{67} - \frac{1}{2}f_{24} + \frac{1}{2}f_{23})y_3y_2^2 \\
\qquad + (\frac{1}{4}f_{85} - \frac{1}{4}f_{42} + \frac{1}{4}f_{41})y_5y_3 + (-2f_{78} - f_{43} + f_{44} - f_{87} + 2f_{58})y_6y_5 \\
\qquad + (2f_{32} + 3f_{12} + 2f_{66} + 5f_{11} + f_{55} + 2f_{31})y_7y_1 + (-\frac{1}{4}f_{41} + \frac{1}{4}f_{42} - \frac{1}{4}f_{85})y_5y_1^2 \\
\qquad + (-\frac{1}{4}f_{23} - \frac{1}{4}f_{67} + \frac{1}{4}f_{24})y_2^2y_1^2 + (\frac{1}{4}f_{67} - \frac{1}{4}f_{24} + \frac{1}{4}f_{23})y_4y_1^2 + 2y_1y_8f_{68} \\
\qquad + (-\frac{1}{2}f_{23} - \frac{1}{2}f_{67} + \frac{1}{2}f_{24})y_5^2 + (f_{65} - f_{22} + 2f_{56} - 2f_{76} + f_{21})y_6y_3 \\
\qquad + (f_{34} + f_{77} + 4f_{13} + 4f_{14} + 3f_{33})y_7y_2 \\
(4(\sigma - d_2) + 2(\lambda - d_1))y_8 + (4f_{42} - 4f_{41} - 4f_{85})y_6^2 + (\frac{1}{4}f_{42} - \frac{1}{4}f_{41} - \frac{1}{4}f_{85})y_3y_2^2 \\
\qquad + (f_{65} - f_{22} + 2f_{56} - 2f_{76} + f_{21})y_6y_5 + (\frac{1}{4}f_{85} - \frac{1}{4}f_{42} + \frac{1}{4}f_{41})y_2^2y_1^2 \\
\qquad + (-\frac{1}{2}f_{41} + \frac{1}{2}f_{42} - \frac{1}{2}f_{85})y_4y_1^2 + (3f_{11} + 4f_{31} + f_{12} + f_{55} + 4f_{32})y_8y_1 \\
\qquad + (-\frac{1}{2}f_{42} + \frac{1}{2}f_{41} + \frac{1}{2}f_{85})y_5^2 + 2y_2y_7f_{86} \\
\qquad + (\frac{1}{4}f_{67} - \frac{1}{4}f_{24} + \frac{1}{4}f_{23})y_5y_2^2 + (5f_{33} + 2f_{13} + 2f_{14} + f_{77} + 3f_{34} + 2f_{88})y_8y_2 \\
\qquad + (-\frac{1}{4}f_{23} - \frac{1}{4}f_{67} + \frac{1}{4}f_{24})y_5y_4 + (-2f_{78} - f_{43} + f_{44} - f_{87} + 2f_{58})y_6y_4
\end{array}\right)$$

We will focus on the 2-dimensional stratum Υ_7, which is the image of the fixed point space of κ. The boundary of Υ_7 is the union of Υ_0, Υ_1, and Υ_4. We consider the restriction to $\overline{\Upsilon_7}$. If we introduce new coordinates ($x = \rho_1 \geq 0$ and $y = \rho_2 \geq 0$) and scale all variables appropriately by powers of ε, then the general system on $\overline{\Upsilon_7}$ becomes

$$\begin{aligned}
\dot{x} &= 2x(\lambda' + a_1y + a_2x + \varepsilon(a_3y^2 + a_4xy + a_5x^2 + \varepsilon(\ldots))) \\
\dot{y} &= 2y(\mu' + b_1x + b_2y + \varepsilon(b_3x^2 + b_4xy + b_5y^2 + \varepsilon(\ldots)))\,,
\end{aligned} \tag{7}$$

where a_i and b_i are constants. The first four constants are given in terms of the cubic terms of f as follows:

$$\begin{aligned}
a_1 &= \tfrac{1}{2}(3f_{13} + f_{14} + f_{57}) & \qquad a_2 &= \tfrac{1}{2}(3f_{11} + f_{12} + f_{55}) \\
b_1 &= \tfrac{1}{2}(3f_{31} + f_{32} + f_{75}) & \qquad b_2 &= \tfrac{1}{2}(3f_{33} + f_{34} + f_{77})
\end{aligned}$$

The next six constants involve the quintic terms of f:

$$
\begin{aligned}
a_3 = {} & \tfrac{1}{2}(f_{134} + 5f_{133} + f_{144} + f_{177} + f_{537} + f_{547}) \\
& +(\tfrac{3}{4}f_{34}f_{14} - \tfrac{3}{4}f_{33}f_{14} + \tfrac{3}{4}f_{77}f_{14} + \tfrac{1}{4}f_{77}f_{13} + \tfrac{1}{4}f_{57}f_{34} - \tfrac{1}{4}f_{57}f_{33} \\
& \quad +\tfrac{1}{4}f_{34}f_{13} - \tfrac{1}{4}f_{33}f_{13} - \tfrac{1}{2}f_{87}f_{58} - \tfrac{1}{2}f_{58}f_{44} + \tfrac{1}{2}f_{58}f_{43} + \tfrac{1}{4}f_{77}f_{57})h_4 \\
& +(\tfrac{1}{4}f_{57}f_{14} - \tfrac{1}{4}f_{57}f_{13} - \tfrac{1}{4}f_{24}f_{23} + \tfrac{1}{4}f_{14}f_{13} - \tfrac{1}{8}f_{13}^2 - \tfrac{1}{8}f_{14}^2 + \tfrac{1}{8}f_{23}^2 \\
& \quad +\tfrac{1}{8}f_{24}^2 + \tfrac{3}{8}f_{57}^2 - \tfrac{3}{8}f_{67}^2 - \tfrac{1}{4}f_{67}f_{24} + \tfrac{1}{4}f_{67}f_{23})h_2 \\
a_4 = {} & +\tfrac{1}{2}(f_{545} + f_{124} + f_{123} + f_{535} + 5f_{113} + f_{157} + f_{517} + f_{527} + f_{114}) \\
& +(-\tfrac{1}{4}f_{75}f_{13} - \tfrac{3}{4}f_{75}f_{14} + \tfrac{3}{4}f_{31}f_{14} - \tfrac{3}{4}f_{32}f_{14} + \tfrac{1}{4}f_{57}f_{31} + \tfrac{1}{2}f_{85}f_{58} \\
& \quad -\tfrac{1}{4}f_{57}f_{32} + \tfrac{1}{2}f_{58}f_{42} + \tfrac{1}{4}f_{31}f_{13} - \tfrac{1}{4}f_{75}f_{57} - \tfrac{1}{4}f_{32}f_{13} - \tfrac{1}{2}f_{58}f_{41})h_3 \\
& +(-\tfrac{1}{8}f_{24}f_{21} - \tfrac{5}{8}f_{13}f_{12} - \tfrac{3}{8}f_{13}f_{11} + \tfrac{3}{8}f_{65}f_{23} - \tfrac{3}{8}f_{67}f_{65} - \tfrac{1}{8}f_{23}f_{22} \\
& \quad -\tfrac{3}{8}f_{65}f_{24} + \tfrac{5}{8}f_{14}f_{12} - \tfrac{5}{8}f_{55}f_{13} - \tfrac{1}{8}f_{67}f_{21} - \tfrac{1}{2}f_{67}f_{56} + \tfrac{1}{8}f_{67}f_{22} \\
& \quad +\tfrac{5}{8}f_{57}f_{12} + \tfrac{3}{8}f_{57}f_{11} + \tfrac{5}{8}f_{57}f_{55} + \tfrac{1}{8}f_{24}f_{22} + \tfrac{1}{2}f_{56}f_{23} + \tfrac{5}{8}f_{55}f_{14} \\
& \quad +\tfrac{1}{8}f_{23}f_{21} - \tfrac{1}{2}f_{56}f_{24} + \tfrac{3}{8}f_{14}f_{11})h_2 \\
& +(-\tfrac{1}{8}f_{65}f_{23} + \tfrac{1}{8}f_{55}f_{13} + \tfrac{1}{8}f_{65}f_{24} + \tfrac{1}{8}f_{24}f_{22} - \tfrac{1}{8}f_{23}f_{22} + \tfrac{3}{8}f_{57}f_{12} \\
& \quad -\tfrac{1}{8}f_{24}f_{21} + \tfrac{1}{8}f_{14}f_{11} + \tfrac{3}{8}f_{57}f_{55} - \tfrac{1}{8}f_{14}f_{12} + \tfrac{1}{8}f_{13}f_{12} + \tfrac{3}{8}f_{67}f_{21} \\
& \quad -\tfrac{3}{8}f_{57}f_{11} - \tfrac{1}{8}f_{13}f_{11} - \tfrac{1}{8}f_{55}f_{14} - \tfrac{3}{8}f_{67}f_{65} + \tfrac{1}{8}f_{23}f_{21} - \tfrac{3}{8}f_{67}f_{22})h_1 \\
a_5 = {} & \tfrac{1}{2}(f_{515} + 5f_{111} + f_{112} + f_{155} + f_{525} + f_{122}) + (-\tfrac{3}{8}f_{11}^2 + \tfrac{5}{8}f_{55}^2 + \tfrac{1}{8}f_{21}^2 \\
& \quad +\tfrac{1}{8}f_{22}^2 + \tfrac{5}{8}f_{12}^2 - \tfrac{3}{8}f_{65}^2 - \tfrac{1}{4}f_{22}f_{21} + \tfrac{5}{4}f_{55}f_{12} - \tfrac{1}{4}f_{65}f_{22} \\
& \quad +\tfrac{1}{4}f_{65}f_{21} - \tfrac{1}{2}f_{65}f_{56} - \tfrac{1}{2}f_{56}f_{22} + \tfrac{1}{2}f_{56}f_{21} - \tfrac{1}{4}f_{55}f_{11} - \tfrac{1}{4}f_{12}f_{11})h_1 \\
b_3 = {} & +\tfrac{1}{2}(f_{312} + 5f_{311} + f_{322} + f_{355} + f_{715} + f_{725}) \\
& +(\tfrac{1}{4}f_{85}f_{42} - \tfrac{1}{4}f_{32}f_{31} - \tfrac{3}{8}f_{75}^2 - \tfrac{1}{4}f_{85}f_{41} + \tfrac{1}{4}f_{75}f_{31} + \tfrac{1}{4}f_{42}f_{41} \\
& \quad +\tfrac{3}{8}f_{85}^2 - \tfrac{1}{8}f_{42}^2 + \tfrac{1}{8}f_{31}^2 + \tfrac{1}{8}f_{32}^2 \\
& \quad -\tfrac{1}{8}f_{41}^2 - \tfrac{1}{4}f_{75}f_{32})h_3 \\
& +(\tfrac{3}{4}f_{55}f_{32} + \tfrac{1}{4}f_{75}f_{55} - \tfrac{3}{4}f_{32}f_{11} + \tfrac{1}{4}f_{75}f_{12} - \tfrac{1}{4}f_{75}f_{11} + \tfrac{1}{4}f_{55}f_{31} \\
& \quad +\tfrac{1}{4}f_{31}f_{12} - \tfrac{1}{4}f_{31}f_{11} + \tfrac{1}{2}f_{76}f_{21} - \tfrac{1}{2}f_{76}f_{65} - \tfrac{1}{2}f_{76}f_{22} + \tfrac{3}{4}f_{32}f_{12})h_1 \\
b_4 = {} & \tfrac{1}{2}(5f_{313} + f_{314} + f_{324} + f_{323} + f_{717} + f_{727} + f_{357} + f_{735} + f_{745}) \\
& +(\tfrac{1}{8}f_{87}f_{42} - \tfrac{3}{8}f_{87}f_{85} - \tfrac{1}{8}f_{33}f_{31} - \tfrac{3}{8}f_{85}f_{44} + \tfrac{3}{8}f_{75}f_{34} + \tfrac{1}{8}f_{43}f_{41} \\
& \quad -\tfrac{1}{8}f_{77}f_{32} - \tfrac{1}{8}f_{43}f_{42} + \tfrac{1}{8}f_{77}f_{31} - \tfrac{1}{8}f_{44}f_{41} - \tfrac{1}{8}f_{34}f_{32} - \tfrac{1}{8}f_{87}f_{41} \\
& \quad +\tfrac{1}{8}f_{34}f_{31} + \tfrac{3}{8}f_{77}f_{75} + \tfrac{1}{8}f_{33}f_{32} + \tfrac{3}{8}f_{85}f_{43} + \tfrac{1}{8}f_{44}f_{42} - \tfrac{3}{8}f_{75}f_{33})h_4 \\
& +(-\tfrac{3}{8}f_{87}f_{41} - \tfrac{1}{8}f_{43}f_{41} + \tfrac{1}{8}f_{44}f_{41} + \tfrac{1}{2}f_{85}f_{78} - \tfrac{5}{8}f_{34}f_{32} + \tfrac{3}{8}f_{87}f_{42} \\
& \quad +\tfrac{5}{8}f_{34}f_{31} + \tfrac{1}{2}f_{78}f_{42} + \tfrac{3}{8}f_{87}f_{85} - \tfrac{1}{8}f_{85}f_{44} - \tfrac{3}{8}f_{33}f_{32} - \tfrac{5}{8}f_{77}f_{32} \\
& \quad +\tfrac{1}{8}f_{43}f_{42} - \tfrac{5}{8}f_{77}f_{75} - \tfrac{5}{8}f_{75}f_{34} - \tfrac{3}{8}f_{75}f_{33} + \tfrac{3}{8}f_{33}f_{31} + \tfrac{5}{8}f_{77}f_{31} \\
& \quad -\tfrac{1}{8}f_{44}f_{42} + \tfrac{1}{8}f_{85}f_{43} - \tfrac{1}{2}f_{78}f_{41})h_3 \\
& +(\tfrac{3}{4}f_{32}f_{14} - \tfrac{1}{4}f_{75}f_{13} + \tfrac{1}{4}f_{31}f_{14} - \tfrac{1}{4}f_{31}f_{13} + \tfrac{1}{4}f_{75}f_{57} - \tfrac{1}{2}f_{76}f_{24} \\
& \quad +\tfrac{1}{2}f_{76}f_{23} + \tfrac{1}{4}f_{75}f_{14} + \tfrac{3}{4}f_{57}f_{32} - \tfrac{3}{4}f_{32}f_{13} + \tfrac{1}{4}f_{57}f_{31} - \tfrac{1}{2}f_{76}f_{67})h_2 \\
b_5 = {} & \tfrac{1}{2}(5f_{333} + f_{377} + f_{344} + f_{737} + f_{747} + f_{334}) + (\tfrac{5}{8}f_{34}^2 - \tfrac{3}{8}f_{87}^2 + \tfrac{5}{8}f_{77}^2 \\
& \quad +\tfrac{1}{8}f_{44}^2 + \tfrac{1}{8}f_{43}^2 + \tfrac{5}{4}f_{77}f_{34} - \tfrac{1}{4}f_{77}f_{33} + \tfrac{1}{4}f_{87}f_{43} - \tfrac{1}{2}f_{78}f_{44} \\
& \quad -\tfrac{3}{8}f_{33}^2 - \tfrac{1}{2}f_{87}f_{78} + \tfrac{1}{2}f_{78}f_{43} - \tfrac{1}{4}f_{44}f_{43} - \tfrac{1}{4}f_{34}f_{33} - \tfrac{1}{4}f_{87}f_{44})h_4
\end{aligned}
$$

J. Menck investigated the system (7) in [29]. In the limit as $\varepsilon \to 0$, and under the conditions $A_1 := a_2 b_2 < 0$ and $A_2 := a_2 b_2 - a_1 b_1 > 0$ a branch of equilibria emerges from the origin inside Υ_1 or Υ_4, depending on the sign of a_2. A secondary symmetry breaking bifurcation arises into the stratum Υ_7. Furthermore, the new branch finally undergoes a Hopf bifurcation. Between consecutive bifurcations, all equilibria are asymptotically stable inside $\overline{\Upsilon_7}$. This scenario persists for small $\varepsilon > 0$.

The Hopf bifurcation is degenerate for $\varepsilon = 0$. But, if ε becomes positive, the direction and stability of the branch of periodic points can be computed. The determining coefficient in the corresponding normal form has the expansion

$$\begin{aligned}\alpha &= 0 + \varepsilon \tfrac{d}{d\varepsilon}\alpha(0) + O(\varepsilon^2) \\ &= \frac{a_2 b_1}{A_2}\Big((b_2 a_2 b_1^2 - b_1 b_2 a_2^2)a_3 + (-b_2^2 a_2^2 + A_2 b_2 a_2 + b_1 b_2^2 a_2 + 2 b_2 b_1 A_2)a_4 \\ &\qquad +(-a_1 b_2^2 a_2 + 2A_2 b_2 a_1 + b_2^3 a_2 - b_2^2 A_2)a_5 \\ &\qquad +(a_1^2 a_2 b_2 - a_1 b_2^2 a_2)b_3 \\ &\qquad +(-b_2 a_1 a_2^2 - b_2^2 a_2^2 + A_2 b_2 a_2 + 2b_1 b_2^2 a_2 + 2 b_2 b_1 A_2)b_4 \\ &\qquad +(a_2^3 b_2 - A_2 a_2^2 - b_1 b_2 a_2^2 + 2A_2 b_1 a_2)b_5\Big)\varepsilon + O(\varepsilon^2)\ .\end{aligned}$$

If $\frac{d}{d\varepsilon}\alpha(0)$ is negative, then the bifurcation is supercritical, and locally the periodic points are stable inside the stratum Υ_7.

It remains to be shown that the solutions in question are also stable with respect to the directions transversal to $\overline{\Upsilon_7}$. To this end, we note that the gradient of all relations of ρ vanishes at points of $\overline{\Upsilon_7}$; thus, the tangent spaces of the orbit space at these points have dimension 8, and all eigenvalues of the differential of the reduced vector field in $\mathbb{R}^8$ are relevant to determine the stability. That differential has in suitable coordinates block structure (cf. (3)).

At the origin there are the following blocks on the diagonal of the differential: A 2×2-matrix corresponding to the quadratic elements of the Hilbert basis, a 4×4-matrix corresponding to the invariants $\rho_3, \ldots, \rho_6$, and another 2×2-matrix corresponding to ρ_7 and ρ_8. By Remark 9, only the first block must be considered. This is already done by analyzing the restriction of the dynamics to $\overline{\Upsilon_7}$.

At points x of the one-dimensional strata Υ_1 or Υ_4 the representation of the isotropy subgroup $\Gamma_x \cong S^1 \times \mathbb{Z}_2$ on the 6-dimensional normal space decomposes into a sum of four non-equivalent irreducible subrepresentations: The first one is trivial and corresponds to the stratum of x, the kernel of the second one-dimensional subrepresentation is S^1, the kernel of the third one is $\mathbb{Z}_2$, and the last one has a trivial kernel. Therefore, a local minimal Hilbert basis on N_x has one linear and three quadratic elements. Thus, here $\dim R = 4$, and the restriction of the differential to R has in suitable coordinates the form

$$\begin{pmatrix} a_1 & * & * & * \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & a_3 & 0 \\ 0 & 0 & 0 & a_4 \end{pmatrix}$$

where $a_i \in \mathbb{R}$ corresponds to the i-th subreprestation.

Since the stratum Υ_7 is two-dimensional, on the diagonal of the differential there are a 2×2-block corresponding to the stratum and a 6×6-block, whose eigenvalues control the stability transversal to the stratum. This 6×6-matrix provides a criterion for the stability of the solutions in the stratum with respect to the transversal directions. If its eigenvalues are negative at the point of the Hopf bifurcation, then we can conclude that the Lyapunov exponent (Floquet exponents) of the periodic trajectories corresponding to those directions are negative near the bifurcation point.

For example, if we demand $f_{23} = f_{24}$ and $f_{41} = f_{42}$, and assume that all other terms are independent of u_x, i.e., $f_{\bullet 2}$, $f_{\bullet 4}$, and $f_{5\bullet}$, …, $f_{8\bullet}$, $f_{5\bullet\bullet}$, …, $f_{8\bullet\bullet}$ vanish, then we have the following sufficient conditions for the asymptotic stability of the branching periodic trajectories.

$$\begin{aligned} f_{11} &> 0 \\ f_{33} f_{11} &< 0 \\ f_{33} f_{11} - f_{31} f_{13} &> 0 \\ -f_{31} f_{33} + f_{13} f_{11} &> 0 \\ (-f_{31} f_{13} + f_{31} f_{33} + f_{11} f_{13} - f_{33} f_{11}) f_{33} f_{11} &> 0 \end{aligned}$$

and $\beta\, b_1 a_2 = -\frac{32}{81}\alpha'(0) A_2 > 0$ with

$$\begin{aligned} \beta := 5\Big(&(-f_{31} f_{13} f_{33}^2 - f_{13} f_{33}^2 f_{11} + 2 f_{13}^2 f_{31} f_{33}) f_{111} \\ &+(-3 f_{31} f_{33}^2 f_{11} + f_{31} f_{13} f_{33} f_{11} + 2 f_{13} f_{31}^2 f_{33}) f_{113} \\ &+(f_{31} f_{33} f_{11}^2 - f_{33} f_{11} f_{31}^2) f_{133} + (-f_{13}^2 f_{11} f_{33} + f_{13} f_{33}^2 f_{11}) f_{311} \\ &+(f_{31} f_{13} f_{33} f_{11} + 2 f_{13} f_{31}^2 f_{33} + f_{33} f_{13} f_{11}^2 - 4 f_{31} f_{33}^2 f_{11}) f_{313} \\ &+(-f_{31} f_{13} f_{11}^2 + 2 f_{13} f_{11} f_{31}^2 - f_{31} f_{33} f_{11}^2) f_{333}\Big) \\ &+\Big(\tfrac{3}{4} f_{11}^3 f_{13} f_{33}^2 - \tfrac{1}{4} f_{11} f_{13} f_{33}^2 f_{21}^2 - \tfrac{1}{4} f_{13} f_{31} f_{33}^2 f_{21}^2 + f_{11}^2 f_{13} f_{31} f_{33}^2 + \tfrac{1}{2} f_{13}^2 f_{31} f_{33} f_{21}^2 \\ &\qquad - \tfrac{1}{2} f_{11} f_{13}^2 f_{31}^2 f_{33} - \tfrac{5}{4} f_{11}^2 f_{13}^2 f_{31} f_{33}\Big) h_1 \\ &+\Big(-\tfrac{3}{2} f_{11}^2 f_{13}^2 f_{31} f_{33} - f_{13}^2 f_{31}^3 f_{33} + 2 f_{11} f_{13} f_{31}^2 f_{33}^2 \\ &\qquad - \tfrac{7}{4} f_{11} f_{13}^2 f_{31}^2 f_{33} + \tfrac{9}{4} f_{11}^2 f_{13} f_{31} f_{33}^2\Big) h_2 \\ &+\Big(\tfrac{1}{4} f_{11} f_{13}^2 f_{31}^2 f_{33} + f_{13}^2 f_{31}^3 f_{33} + \tfrac{3}{2} f_{13} f_{31}^3 f_{33}^2 - 3 f_{11} f_{31}^2 f_{33}^3 \\ &\qquad + \tfrac{3}{4} f_{11}^2 f_{13} f_{31} f_{33}^2 - \tfrac{1}{2} f_{11} f_{13} f_{31}^2 f_{33}^2\Big) h_3 \\ &+\Big(f_{11} f_{31}^2 f_{33}^3 - \tfrac{1}{2} f_{13} f_{31}^3 f_{33}^2 - \tfrac{1}{4} f_{11}^2 f_{31} f_{33} f_{43}^2 + \tfrac{1}{2} f_{11} f_{13} f_{31}^2 f_{43}^2 \\ &\qquad + \tfrac{3}{4} f_{11}^2 f_{31} f_{33}^3 - \tfrac{1}{4} f_{11}^2 f_{13} f_{31} f_{43}^2 - \tfrac{5}{4} f_{11} f_{13} f_{31}^2 f_{33}^2\Big) h_4 , \end{aligned}$$

where $h_1 = \frac{1}{3d_1 - 39 - \lambda_0}$, $h_2 = \frac{1}{-39 + 4d_1 - d_2 - \sigma_0}$, $h_3 = \frac{1}{d_1 + 39 + \lambda_0 - 4d_2}$, $h_4 = \frac{1}{3d_2 - \sigma_0 - 39}$. For example, this is fulfilled, if $f_{11} = 1$, $f_{13} = -\frac{3}{4}$, $f_{31} = 2$, $f_{33} = -1$, and all other coefficients vanish. In this situation the equilibria of the branches in Υ_4 and Υ_7 are asymptotically stable in the full space $\mathbb{R}^8$ up to the bifurcations. There are similar conditions in the case $f_{11} < 0$.

By the general theory we finally conclude that under the previous conditions the following bifurcations occur in the original problem (6) near the critical parameter value considered. The trivial solution becomes unstable in Z, and a branch of stable, invariant 2-tori emerges, which bifurcate again to stable 3-dimensional invariant orbits. These relative equilibria become unstable at a "Hopf point", where finally a branch of stable 4-dimensional invariant manifolds of relative periodic points emerges.

8 Reconstruction of eigenvalues

This section is devoted to the question, whether the eigenvalues of $DP(0)$ can be reconstructed with the aid of the spectrum of $DQ(0)$, where $P : V \to V$ is a Γ-equivariant map with equilibrium 0 and $Q = \mathcal{J}_{\mathsf{m}}(P)$ is the reduced map. By Theorem 6, the eigenvalues of $DQ(0)$ are products of the eigenvalues of $DP(0)$. The fixed point space V^Γ is invariant with respect to $DP(0)$, and since $D\rho(0)|_{V^\Gamma}$ is one-to-one, the eigenvalues of $DP(0)$ which have an eigenvector in V^Γ appear on the orbit space. Therefore, an eigenvalue of $DQ(0)$ with an eigenvector tangent to the stratum of $\rho(0)$ is also an eigenvalue of $DP(0)$.

Due to the block structure (3) of $DQ(0)$ there is the invariant subspace R of T_0Y which is dual to space of the linear and quadratic invariants. We will only use the restriction $DQ(0)|_R$ to reconstruct the eigenvalues of $DP(0)$. It may happen that the blocks A_1 and A_2 corresponding to the spaces of linear and quadratic invariants, respectively, have a common eigenvalue, then its multiplicity depends on the block above A_2, which is determined by nonlinear terms of P. As we do not care so much about multiplicity of eigenvalues here, we occasionally assume that the right upper block of the matrix $DQ(0)|_R$ is zero.

Remark 29. Let $V^\Gamma \oplus \bigoplus_{i=1}^{j} V_i$ be a decomposition of V into irreducible subrepresentations. Let $\rho_1, \ldots, \rho_k$ be a multihomogeneous minimal Hilbert basis with respect to this decomposition. Let α be the multidegree of a quadratic element ρ_i. Then either $\alpha_j = 2\delta_{j,k}$ for some k or $\alpha_j = \delta_{j,k} + \delta_{j,l}$ for some pair (k,l), where V_k and V_l are equivalent. This is because $\operatorname{grad}\rho_i$ is a Γ-equivariant linear map.
Thus, each linear and quadratic element of the Hilbert basis belongs to a unique isotypical component. Since R is dual to the space of these invariants (in the sense of Remark 18), the subspace R of T_0Y decomposes uniquely into $\bigoplus_i R_{X_i}$, where R_{X_i} corresponds to the isotypical component X_i and is $DQ(0)$-invariant.
The map $\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} : R \to R$ keeps this decomposition invariant, because there are no equivariant linear maps between nonequivalent representations but the trivial one. Of course, this is not correct for $DQ(0)|_R$, in general.

Remark 30. Using the techniques of Sect. 4 the last remark can be carried over to relative equilibria.

As Example 11 shows, there are eigenvalues in the tangent space of the orbit space which have no meaning to the original system P. Especially, these eigenvalues may be critical, even if $DP(0)$ has no critical eigenvalue.

The generalized eigenspace of the stable (unstable) eigenvalues is tangent to the stable (unstable) manifold. A Hilbert map maps the stable (unstable) manifolds into stable (unstable) manifolds on the orbit space.

Proposition 31. *If $P \in C^{qn}_\Gamma(V,V)$ is a diffeomorphism, and $Q := \mathcal{J}_{\mathsf{m}}(P)$ the reduced map, let $W^s(\Gamma x)$ ($W^u(\Gamma x)$) be the stable (unstable) manifold of the invariant orbit Γx of P, and $W^s(y)$ ($W^u(y)$) be the stable (unstable) manifold of the equilibrium $y := \rho(x)$ of Q.*
Then $\rho(W^s(\Gamma x)) = W^s(y) \cap Y$ and $\rho(W^u(\Gamma x)) = W^u(y) \cap Y$.

Proof. Let $z \in W^s(\Gamma x)$, then the distance $d(P^t(z), \Gamma x)$ of the iterates of x to the invariant orbit shrinks exponentially for $t \to \infty$. Also $d(Q^t(\rho(z)), y)$ shrinks exponentially for $t \to \infty$, i.e., $\rho(z) \in W^s(y)$. Also the converse holds. □

This suggests that only eigenvalues whose eigenspaces are 'tangent' to Y in a narrower sense should be used to determine the noncritical eigenvalues at a relative equilibrium. Here, a vector v is 'tangent' to $W^s(y) \cap Y$ means that there is a path in Y starting at y in direction v. This concept leads to the tangent cone.

The tangent cone of the orbit space The *tangent cone* C_yY of a set $Y \subset \mathbb{R}^k$ at $y \in Y$ is the limit of the secants of Y through y:

$$s \in C_yY :\Longleftrightarrow \exists \lambda_i > 0 : \exists y_i \in Y : y_i \to y \wedge \lambda_i(y_i - y) \to s \qquad \text{as} \quad i \to \infty$$

C_yY is a union of rays emerging from y.

First, we consider the tangent cone of the embedded orbit space $Y = \rho(V)$ at 0, where ρ is a minimal Hilbert map. The representation space V decomposes Γ-invariantly into the fixed point space $V_1 = V^\Gamma$ and the orthogonal complement V_2. Let ρ_1 be a minimal Hilbert map on V_1, i.e., coordinates on V_1, and (ρ_2, ρ_3) a minimal homogeneous Hilbert map on V_2, where ρ_2 is a quadratic map and ρ_3 contains the higher order terms. Then $\rho = (\rho_1, \rho_2, \rho_3)$ is a Hilbert map on V, and $Y = \rho_1(V_1) \times (\rho_2, \rho_2)(V_2) = V_1 \times (\rho_2, \rho_3)(V_2)$.

Proposition 32 ([23]). *The tangent cone of Y at 0 is given by $C_0Y = V_1 \times \rho_2(V_2) \times \{0\}$.*

Proof. In the last paragraph we proved that $C_0Y = V_1 \times C_0(\rho_2, \rho_3)(V_2)$. We proceed by showing $C_0(\rho_2, \rho_3)(V_2) = C_0\rho_2(V_2) \times \{0\}$. Take $s = (s^{(2)}, s^{(3)}) \in C_0(\rho_2, \rho_3)(V_2)$. There are sequences $\lambda_i > 0$ and $y_i \in (\rho_2, \rho_3)(V_2)$ with $y_i \to 0$ and $\lambda_i y_i \to s$.

Choose $x_i \in V_2$ with $y_i = (\rho_2(x_i), \rho_3(x_i)) =: (y_i^{(2)}, y_i^{(3)})$. x_i converges to 0. Since $R : x \mapsto \langle x, x \rangle$ is Γ-invariant, there is linear function g with $R = g \circ \rho_2$. If the component r of ρ_3 has degree $d > 2$, then $|r(x_i)| \leq CR(x_i)^{d/2} = Cg(y_i)^{d/2} \leq C'\|y_i\|^{d/2}$, where C and C' are suitable constants. This yields $|\lambda_i r(x_i)| \leq C'\|\lambda_i y_i\| \, \|y_i\|^{d/2-1} \to C'\|s\|0 = 0$. Therefore, $s^{(3)} = 0$.

We have $\lambda_i y_i^{(2)} = \rho_2(\sqrt{\lambda_i} x_i)$. Since ρ_2 is proper, $\sqrt{\lambda_i} x_i$ is bounded, and has a limit $v \in V_2$, eventually pass to a subsequence. Hence, $\rho_2(v) = \rho_2(\lim \sqrt{\lambda_i} x_i) = \lim \lambda_i \rho_2(x_i) = s^{(2)}$, and $s \in \rho(V_2) \times \{0\}$. □

Now we consider the tangent cone at an arbitrary point $y \in Y$. Let $x \in V$ such that $\rho(x) = y$. We regard the tangent cone as a subset of $T_y\mathbb{R}^k$. To achieve this we identify the tangent space $T_y\mathbb{R}^k$ with $\mathbb{R}^k$ in the canonical way. Obviously, the tangent cone C_yY contains the tangent space $T_y\Upsilon_y$ of the stratum Υ_y of y. And it is a subset of the tangent space T_yY of the orbit space: Let r be a relation of ρ, and write $r(z) = Dr(y)(z-y) + r_2(z-y)$. Let $s \in C_yY$ and $y_i \in Y$ be a sequence with limit y, and $\lambda_i > 0$ a sequence with $\lambda_i(y_i - y) \to s$. Setting $d_i := (y_i - y) - s/\lambda_i$, we have

$$\begin{aligned} 0 &= r(y_i)\lambda_i = r(y + s/\lambda_i + d_i)\lambda_i = Dr(y)s + Dr(y)d_i\lambda_i + r_2(s/\lambda_i + d_i)\lambda_i \\ &= Dr(y)s + Dr(y)d_i\lambda_i + o(1/\lambda_i) + o(d_i) \longrightarrow Dr(y)s\,, \end{aligned}$$

because $d_i\lambda_i \to 0$.

Proposition 33. *Let ρ and σ be two Hilbert maps for the same representation. Denote the embedded orbit spaces by $Y_\rho = \rho(V)$ and $Y_\sigma = \sigma(V)$, respectively. There is a polynomial Λ with $\sigma = \Lambda \circ \rho$. The isomorphism $D\Lambda(\rho(x)) : T_{\rho(x)}Y_\rho \to T_{\sigma(x)}Y_\sigma$ induces a bijection between $C_{\rho(x)}Y_\rho$ and $C_{\sigma(x)}Y_\sigma$, $x \in V$.*
For $x \in V$ and $y := \rho(x)$, the isomorphism $DH_x(0) : T_0Y_x \to T_yY$ from Lemma 16 induces a bijection between the tangent cones C_0Y_x and C_yY, where Y_x is the local orbit space.

Proof. We proof the second statement. Let $s \in C_0Y_x$, $y_i \in Y_x$, and $\lambda_i > 0$ with $y_i \to 0$ and $\lambda_i y_i \to s$ for $i \to \infty$, and $d_i := (y_i - y) - s/\lambda_i$, then

$$\begin{aligned} \lambda_i(H_x(y_i) - y) &= \lambda_i(H_x(s/\lambda_i + d_i) - y) \\ &= DH_x(0)s + DH_x(0)\lambda_i d_i + o(1/\lambda_i) + o(d_i) \longrightarrow DH_x(0)s\,. \end{aligned}$$

Thus, $DH_x(0)$ maps C_0Y_x into C_yY. The converse is shown in the same way. □

Remark 34. For a given point $x \in V$ write $V_2 := (V^{\Gamma_x})^\perp$. Expand ρ at x according to Taylor, and denote the linear part of the expansion by $\rho_\mathrm{l} = D\rho(x)$ ($V_2 = \ker\rho_\mathrm{l}$), the quadratic part of $\rho|_{V_2}$ by ρ_q, and the other terms by ρ_r. Then $\rho = \rho(x) + \rho_\mathrm{l} + \rho_\mathrm{q} + \rho_\mathrm{r}$, and $C_{\rho(x)}Y = \rho_\mathrm{l}(V) + \rho_\mathrm{q}(V) \subset T_{\rho(x)}Y$ (see Propositions 32 and 33).
Moreover, for $x = 0$, $C_0Y = \rho_\mathrm{l}(V) + \sum_j \rho_\mathrm{q}(X_j)$, where $X_1, \ldots$ are the isotypical components of V. Note that $\rho_\mathrm{l}(V) = \rho_\mathrm{l}(V^\Gamma)$ and $\rho_\mathrm{q}(V^\Gamma) = \{0\}$.

Remark 35. If the Hilbert basis $\rho_1, \ldots, \rho_k$ is homogeneous and ordered by degree, then the differential of the reduced map has the form (3). The diagonal part is obtained by reducing $DP(0)$:

$$S := D\mathcal{J}_\mathrm{m}(DP(0)) = \begin{pmatrix} A_1 & 0 \\ 0 & \ddots \end{pmatrix}$$

Moreover, $(\rho_\mathrm{l} + \rho_\mathrm{q}) \circ DP(0) = S \circ (\rho_\mathrm{l} + \rho_\mathrm{q})$.

Lemma 36. *Suppose that $V_1 \subset V$ is invariant with respect to both the group action and $DP(0)$. Then there is a subspace $R' \subset R_0$ which is S-invariant, and contains $(\rho_l + \rho_q)(V_1) \subset C_0 Y$.*
If V_1 is irreducible, then we can choose R' to be one-dimensional. If V_1 is the sum of two equivalent nontrivial irreducible representations of type $\mathbb{R}$, then a three-dimensional R' can be chosen.
If V_1 being the sum of m equivalent nontrivial irreducible representations has a cyclic element v, i.e., $V_1 := \mathrm{span}\{\Gamma v, \dots, \Gamma DP(0)^{m-1} v\}$, then the subspace $R'' = \mathrm{span}\{S^i \rho_q(v) \mid i \in \mathbb{N}_0\} \subset R'$ is at least m-dimensional.

Proof. If X is a isotypical component, then $X \cap V_1$ is $DP(0)$-invariant. Therefore, we can assume V_1 to be contained in an isotypical component. If X is the fixed point space V^Γ, then nothing remains to show, because $D\rho(0)$ is an isomorphism between V^Γ and the tangent space of the stratum of the origin.

Now, suppose V_1 lies in a nontrivial isotypical component. There is a Γ-invariant complement of V_1, say V_2. Remark 19 applies to this situation. The image $\rho_q(V_1)$ is a subset of $R' := (L_{2,0}/J_0^2)^*$: Choose a multihomogeneous Hilbert basis $\sigma_1, \dots, \sigma_k$ ordered in such a way that $\sigma_i \in L_{2,0}$, if and only if $i \le l$, for some l. For $v \in V_1$ we have $\sigma_i(v) = 0$ provided that $i > l$ and $\deg \sigma_i = 2$. It follows $\sigma_q(v) = (a_1, \dots, a_l, 0, \dots, 0)$, where $a_i \in \mathbb{R}$ and, therefore, $\sigma_q(v) \in R'$. A change to the Hilbert map ρ does not affect the statement.

If V_1 is irreducible, then $l = 1$. Therefore, $\dim L_{2,0}/J_0^2 = 1$. In the second case, we have $l = 3$.

With respect to a certain basis of $V_1 = \mathrm{span}\{\Gamma v\} \oplus \mathrm{span}\{\Gamma DP(0)v\} \oplus \cdots$ we obtain the block matrix $\begin{pmatrix} 0 & \cdots & 0 & C \\ \mathrm{id} & \ddots & \vdots & * \\ & \ddots & 0 & \vdots \\ 0 & & \mathrm{id} & * \end{pmatrix}$ for $DP(0)|_{V_1}$, where $\det C \ne 0$. Let $\sigma_i \in L_{2,0}$ be the radius on $\mathrm{span}\{\Gamma DP^{i-1} v\}$, $i = 1, \dots, m$. Then σ_1 is a cyclic element of a subspace of $L_{2,0}$ with dimension $\ge m$. If at least one of the $*$'s stands for a non-zero matrix, then this space has dimension $> m$. The dual $R'' \subset R'$ of this space has the same dimension and possesses $\sigma_q(v)$ as a cyclic element. □

Theorem 37. *Let ρ be a homogeneous Hilbert basis, and P and Q be as in (2). Suppose $P(0) = 0$. Let S be as in Remark 35. Let λ be an eigenvalue of $DP(0)$.*

1. *If the eigenspace of λ contains a trivial subrepresentation, then λ is an eigenvalue of S, and there is an eigenvector of λ tangent to the stratum of $\rho(0)$.*
2. *If λ is real, and there is a nontrivial representation in the eigenspace of λ, then $\mu := \lambda^2$ is an eigenvalue of S with an eigenvector in the tangent cone $C_0 Y$.*

3. *If λ is not real, and there is an irreducible subrepresentation of type $\mathbb{C}$ or $\mathbb{H}$ in the eigenspace of λ, then $\mu := \lambda\bar{\lambda}$ is an eigenvalue of S, and there exists an eigenvector of μ in $C_0 Y$.*
4. *If the eigenspace of $\lambda \notin \mathbb{R}$ contains a nontrivial irreducible subrepresentation of type $\mathbb{R}$, then S possesses the eigenvalues λ^2, $\lambda\bar{\lambda}$ and $\bar{\lambda}^2$ and there are eigenvectors w_1, w_2, and w_3 corresponding to these eigenvalues such that the real part of* $\operatorname{span}\{w_1, w_2, w_3\}$ *intersects the tangent cone $C_0 Y$ nontrivially.*

Proof. 1. Since $D\rho(0) : V^\Gamma \to T_0\Upsilon_0$ is an isomorphism, we obtain the claim from $DQ(0)D\rho(0) = D\rho(0)DP(0)$.

2. Suppose V_1 is a nontrivial irreducible subrepresentation in the eigenspace of λ. By Lemma 36 there is a one-dimensional subspace R' of R_0, which contains $\rho_{\mathrm{q}}(V_1) \subset C_0 Y$. Since R' is an invariant space for S, it must lie in an eigenspace. According to the proof of Theorem 6, the corresponding eigenvalue of S is $\mu := \lambda^2$.

3. Let W_1 be an irreducible (complex) subrepresentation of type $\mathbb{C}$ or $\mathbb{H}$ in the eigenspace of λ. Of course, $\bar{\lambda}$ is an eigenvalue of $DP(0)$, and its eigenspace contains $\overline{W_1}$. The real part V_1 of $W_1 \oplus \overline{W_1}$ is an irreducible subrepresentation. As above there is an S-invariant line R'. Note that there is a Γ-invariant quadratic polynomial on V_1 which is of degree 1 on W_1 and $\overline{W_1}$. Therefore, the eigenvalue of S on R' is $\lambda\bar{\lambda}$.

4. Denote the mentioned representation by W_1. Then the real part V_1 of $W_1 \oplus \overline{W_1}$ is the sum of two equivalent irreducible representations $V_1^1 \oplus V_1^2$. Let $L_{2,0}$ the space of Γ-invariant polynomials of multidegree $(2,0)$ as in Remark 19. By Lemma 36 we have a three-dimensional S-invariant subspace $R' = (L_{2,0}/J_0^2)^*$ of R.

$L_{2,0} \otimes_{\mathbb{R}} \mathbb{C}$ contains polynomials $\sigma_1, \ldots, \sigma_3$ with multidegree $(2,0)$, $(1,1)$, and $(0,2)$ with respect to $W_1 \oplus \overline{W_1}$. According to the proof of Theorem 6 there are eigenvalues λ^2, $\lambda\bar{\lambda}$, and $\bar{\lambda}^2$ of S with eigenvectors in $R' \otimes_{\mathbb{R}} \mathbb{C}$. Therefore, these eigenvectors generate $R' \otimes_{\mathbb{R}} \mathbb{C}$. □

Remark 38. If we replace S by $DQ(0)$, the statements of the theorem remains true, due to the structure of the tangent cone (Remark 34).
Note that if A_1 and A_2 have a common eigenvalue, then all of its eigenvectors may be tangent to the stratum of the origin. In part 4, then the intersection of the real part of $\operatorname{span}\{w_1, w_2, w_3\}$ and the tangent cone is contained in $\mathbb{R}w_2$.

Example 39. Consider the action of $\mathbb{Z}_2$ on $\mathbb{R}^3$ given by $-\operatorname{id}(x, y, z) = (x, -y, -z)$, and the equivariant map $P(x, y, z) = (2x + y^2 + z^2, y - z, y + z)$. The eigenvalues of $DP(0)$ are 2 and $1 \pm \mathrm{i}$. If we choose the Hilbert basis x, y^2, z^2, yz, then $DQ(0)$ has the shape

$$\begin{pmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & -2 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & -1 & 0 \end{pmatrix}$$

and the eigenvalues 2, 2i, and $-2\mathrm{i}$ with eigenvectors $(1,0,0,0)$, $(0,\mathrm{i},-\mathrm{i},1)$, and $(0,-\mathrm{i},\mathrm{i},1)$, respectively. The first eigenvalue is double. The span of the eigenvectors intersects the tangent cone $C_0Y = Y = \{(u_1,u_2,u_3,u_4) \mid u_2u_3 = u_4^2 \wedge u_2, u_3 \geq 0\}$ at the line $\mathbb{R}(1,0,0,0)$.

The eigenvalue 2 of S has the linear independent eigenvectors $(1,0,0,0)$ and $(0,1,1,0)$. The intersection of $\mathrm{span}\{(a,1,1,0),(0,\mathrm{i},-\mathrm{i},1),(0,-\mathrm{i},\mathrm{i},1)\}$, $a \in \mathbb{R}$, and C_0Y contains no eigenvector of S.

We state a converse of the last theorem, which allows to reconstruct important facts about the eigenvalues of $DP(0)$.

Proposition 40. *Resume the situation of the last Theorem. If μ is an eigenvalue of $DQ(0)$, and the eigenvector w of μ lies in the complexification of $T_0\Upsilon_0$, the tangent space of the stratum of 0, then $DP(0)|_{V^\Gamma}$ has the eigenvalue μ.*

Proof. This follows from the proof of part 1 of Theorem 37. □

Theorem 41. *Resume the setting of Theorem 37. Let μ be a real eigenvalue of S with (real) eigenvector w. Assume $w \notin T_0\Upsilon_0$.*

1. *If w is contained in the tangent cone C_0Y, then $DP(0)$ has an eigenvalue λ with $\lambda\bar{\lambda} = \mu$.*
2. *Suppose $\mathbb{R}w \cap C_0Y = \{0\}$, and assume μ_1 and $\bar{\mu}_1$ to be a pair of complex eigenvalues of S with $|\mu| = |\mu_1|$. Denote their eigenvectors by w, w_1, and $\bar{w}_1$, respectively, and assume that the space W generated by w, w_1, and $\bar{w}_1$ has a nontrivial intersection with the tangent cone. Then $DP(0)$ has eigenvalues λ and $\bar{\lambda}$ with $\lambda\bar{\lambda} = \mu$ and $\lambda^2 = \mu_1$.*

Proof. 1. w lies in the space $R \subset T_0Y$ (see Remark 9 and Proposition 32). Since R decomposes into S-invariant subspaces R_X (Remark 29), the projection w_X of w onto R_X is either the zero vector or an eigenvector of μ. w_X lies in the tangent cone C_0Y because of the product structure of C_0Y (Remark 34). Thus, we can assume $w \in R_X$, where X is not the fixed point space. There exists a point $v \in X$ with $\rho_{\mathrm{q}}(v) = w$. The space $V_1 := \mathrm{span}\,\Gamma v$ is an irreducible subrepresentation. We will show that V_1 is invariant with respect to $DP(0)$.

Let $V = V_1 \oplus \bigoplus_{i>1} V_i$ be a decomposition into irreducible subrepresentations. Choose a multihomogeneous Hilbert map σ with respect to $\bigoplus_i V_i$. Then there is one component, say σ_1, with multidegree $(2,0,\ldots)$. There is a polynomial Λ with $\sigma = \Lambda \circ \rho$, which defines a change of coordinates of the orbit space. We have $\sigma_{\mathrm{q}} = D\Lambda(0)\rho_{\mathrm{q}}$, and get $D\Lambda(0)w = \sigma_{\mathrm{q}}(v) = (\sigma_1(v), 0, \ldots) =: w'$, because the quadratic components of σ except of σ_1 vanish on V_1 (Remark 29). Furthermore:

$$(\sigma_{\mathrm{l}} + \sigma_{\mathrm{q}})(x) \in \mathbb{R}w' \Longleftrightarrow x \in V_1$$

We show that V_1 is invariant with respect to $DP(0)$. The map $S' := DA(0) S DA(0)^{-1}$ has the eigenvalue μ with eigenvector w'. Remark 35 yields for $x \in V_1$

$$(\sigma_\mathrm{l} + \sigma_\mathrm{q})(DP(0)x) = S' \circ (\sigma_\mathrm{l} + \sigma_\mathrm{q})(x) = S'(cw') = \mu c w' = \mu(\sigma_\mathrm{l} + \sigma_\mathrm{q})(v) \ ,$$

where c is a real constant. Therefore, $DP(0)x \in V_1$. This infers that $DP(0)|_{V_1}$ has either only one eigenvalue λ or a pair of eigenvalues λ and $\bar{\lambda}$. From the proof of Theorem 6 we conclude that $\mu = \lambda\bar{\lambda}$.

2. We can assume $W \subset R_X \otimes_\mathbb{R} \mathbb{C}$, where $X \neq V^\Gamma$ is an isotypical component of V. There is a point $v \in X$ with $w := \rho_\mathrm{q}(v) \in W$.

Let V_1 by the smallest vector space which contains v and is invariant with respect to Γ and $DP(0)$. V_1 is the sum of m equivalent irreducible subrepresentations. We show $m = 2$, and that the type of the representation is real. By Lemma 36, there is an m-dimensional space R'' generated by the iterates of w. Therefore, $m = \dim R'' \leq \dim W = 3$. If $m = 1$ or $m = 3$, then $DP(0)|_{V_1}$ has a real eigenvalue. Since $C_0 Y \cap W$ does not contain any eigenvector of $S|_W$, neither $DP(0)|_{V_1}$ can have a real eigenvector nor the type of V_1 can be $\mathbb{C}$ or $\mathbb{H}$ (cf. Theorem 37 parts 2 and 3). Therefore, we have that $m = 2$, V_1 is of real type, and $DP(0)|_{V_1}$ has a pair of complex eigenvalues λ and $\bar{\lambda}$. Due to Theorem 6 either μ_1 or $\bar{\mu}_1$ must be the square of λ. □

Remark 42. Due to Theorem 37 all eigenvalues of $DP(0)$ can be discovered by Proposition 40 and Theorem 41.
If we use $DQ(0)$ instead of S, the conditions in Theorem 41 are stronger, but the assertions remain true. If there are common eigenvalues in the blocks A_1 and A_2, it may appear that not all eigenvalues of $DP(0)$ can be discovered using $DQ(0)$: Resume Example 39, considering the eigenvectors of $DQ(0)$ we cannot conclude that $DP(0)$ has the pair of eigenvalues $1 \pm \mathrm{i}$ using part 2 of Theorem 41. However, this conclusion is possible, if we consider the eigenvectors of S.
Let B be an arbitrary complement of $T_0\Upsilon_0$ in $T_0 Y$ with projection $\Pi : T_0 Y \to B$. Define $S' = DQ(0)|_{T_0\Upsilon_0} + \Pi \circ DQ(0)$. The Theorems 37 and 41 remain true, if we replace S by S'.
We get the same result, if we consider generalized eigenvectors in place of genuine eigenvectors.

Example 43. Consider the matrix $\begin{pmatrix} a & -b & 0 & 0 \\ b & a & 0 & 0 \\ 0 & 0 & c & -d \\ 0 & 0 & d & c \end{pmatrix}$ with eigenvalues $\lambda_{1,2} = a \pm \mathrm{i}b$ and $\lambda_{3,4} = c \pm \mathrm{i}d$, which is equivariant with respect to the diagonal action of $\mathrm{SO}(2)$ on $\mathbb{R}^4$. The orbit space is given by the relations $\rho_1\rho_2 = \rho_3^2 + \rho_4^2$ and $\rho_1 \geq 0$. Since the Hilbert basis is quadratic, we have $C_0 Y = Y$. The eigenvalues of the reduced map are $\mu_1 = \lambda_1\lambda_2 = a^2 + b^2$, $\mu_2 = \lambda_3\lambda_4 = c^2 + d^2$, and $\mu_{3,4} = \lambda_{1,2}\lambda_{3,4} = ac + bd \pm \mathrm{i}(ad - bc)$ with eigenvectors $(1,0,0,0)$, $(0,1,0,0)$, and $(0,0,1,\pm\mathrm{i})$, respectively.

Even if $\mu_3 = \mu_4 \in \mathbb{R}$, none of their eigenvectors lies in the tangent cone. Applying Theorem 41 we can deduce that P has an eigenvalue λ or a pair of eigenvalues λ and $\bar{\lambda}$ with $|\lambda|^2 = a^2 + b^2$ and another eigenvalue μ or another pair of eigenvalues μ and $\bar{\mu}$ with $|\mu|^2 = c^2 + d^2$.

If we consider the symmetry of the subgroup $\mathbb{Z}_2 = \{\mathrm{id}, -\mathrm{id}\}$, then a minimal Hilbert basis consists of 10 quadratic elements: x_1^2, x_2^2, x_1x_2, x_3^2, x_4^2, x_3x_4, x_1x_3, x_2x_4, x_1x_4, and x_2x_3.

The reduced system has the eigenvalues $\mu_1 = \lambda_1^2$, $\mu_2 = \lambda_2^2$, $\mu_3 = \lambda_1\lambda_2$, $\mu_4 = \lambda_3^2$, $\mu_5 = \lambda_4^2$, $\mu_6 = \lambda_3\lambda_4$, $\mu_7 = \lambda_1\lambda_3$, $\mu_8 = \lambda_1\lambda_4$, $\mu_9 = \lambda_2\lambda_3$, and $\mu_{10} = \lambda_2\lambda_4$. Suppose we have computed the tangent cone and the eigenvectors, then we can conclude by Theorem 41 that the original system has two pairs of eigenvalues λ, $\bar{\lambda}$, μ, and $\bar{\mu}$ with $\lambda^2 = \mu_1$ and $\mu^2 = \mu_4$.

The eigenvalues $\mu_7, \ldots, \mu_{10}$ have the same modulus. If the arguments of λ_1 and λ_3 are equal, and $\lambda_1 \notin \mathbb{R}$, then $\mu_8 = \mu_9 \in \mathbb{R}$, and (μ_7, μ_{10}) is a complex pair of eigenvalues. Obviously, both triples (μ_7, μ_8, μ_{10}) and (μ_7, μ_9, μ_{10}) satisfy the first condition of part 2 of Theorem 41. However, for all triples of eigenvectors corresponding to these eigenvalues the second condition fails.

The same situation occurs, if $|\lambda_1| = |\lambda_3|$. Then we have $\mu_3 = \mu_6 \in \mathbb{R}$ and four pairs of complex conjugated eigenvalues (μ_1, μ_2), (μ_4, μ_5), (μ_7, μ_{10}), and (μ_8, μ_9) on the orbit space; all of them have the modulus μ_3. Therefore, any pair together with either μ_3 or μ_6 satisfy the first condition of part 2 of the theorem, but the second condition can only be satisfied for the triples (μ_1, μ_2, μ_3) and (μ_4, μ_5, μ_6).

Using Proposition 24 we adopt these results for an arbitrary equilibrium of Q on Y. The following theorem is an analogue to Theorem 41.

Theorem 44. *Let P and Q be as in (2). Assume Γx to be an invariant orbit of P. Then $y := \rho(x)$ is an equilibrium of Q. Let $\widetilde{P}$ and a as in Proposition 24. Let B be a complement of $T := T_y\Upsilon_y$ in T_yY with projection Π, where Υ_y is the stratum of y. Define $S = DQ(y)|_{T_y\Upsilon_y} + \Pi \circ DQ(y)$.*
Let μ be an eigenvalue of $S|_{R_y}$ (or $S|_{T_yY}$) with eigenvector w.

1. *If $w \in T \otimes_{\mathbb{R}} \mathbb{C}$, then $D\widetilde{P}(0)$ has the eigenvalue μ^a.*
2. *If $\mu \in \mathbb{R}$ and $w \in C_yY \cap B$, then $D\widetilde{P}(0)$ has an eigenvalue λ with $\lambda\bar{\lambda} = \mu^a$.*
3. *Suppose $\mu \in \mathbb{R}$ and that S has the complex eigenvalues μ_1 and $\bar{\mu}_1$ such that $|\mu| = |\mu_1|$ and the eigenvectors w, w_1 and $\bar{w}_1$, respectively, generate a space W which has a nontrivial intersection with $C_yY \cap B$. Then $D\widetilde{P}(0)$ has eigenvalues λ and $\bar{\lambda}$ with $\lambda^2 = \mu_1^a$ and $\lambda\bar{\lambda} = \mu^a$.*

Corollary 45. *The invariant orbit is normally hyperbolic, if and only if for every eigenvector w of 1 of $DQ(y)$ neither $\mathbb{R}w \cap C_yY \neq \{0\}$ nor there is a critical eigenvalue $\mu_1 \notin \mathbb{R}$ of $DQ(y)$ with eigenvector w_1 such that* $\mathrm{span}\{w, w_1, \bar{w}_1\} \cap C_yY \neq \{0\}$.

Proof. Γx is normally hyperbolic, if and only if $D\widetilde{P}(0)$ has no critical eigenvalues. It follows immediately from the above theorem that $D\widetilde{P}(0)$ has no critical eigenvalue, if and only if all eigenvectors corresponding to the eigenvalue 1 of S satisfy the condition stated above.

However, this condition holds for $DQ(y)$, if and only if it holds for S. Because if the case handled in Remark 38 arises, then A_1 has already a critical eigenvalue. □

The statements of this section are also valid for vector fields. In that context, Kœnig derived a similar criterion for hyperbolicity in [23]: Γx is normally hyperbolic, if and only if the center space of $DQ(y)$ and the tangent cone have a trivial intersection.

9 Reductive, non-compact groups

Finally, we drop the assumption of compactness of Γ. To guarantee the existence of a Hilbert basis, the existence of slices, and the validity of the slice theorem and Proposition 2 we just need to assume that the group Γ is a reductive real algebraic group (see [34]), i.e., all rational representations of Γ are completely reducible.

However, then orbits are not necessarily compact any more, and it is possible that an orbit is contained in the closure of another one. Therefore, we cannot expect that a Hilbert map separates the orbits. Since each orbit contains an unique closed orbit, the space of closed orbits, called the algebraic quotient, rather than the orbit space should be embedded by a Hilbert map. Again, a set of inequalities exists which describes the embedded set ([6]). Although a Hilbert map separates the closed orbits of the complexified group $\Gamma_{\mathbb{C}}$, this is not valid for the orbits of Γ themselves. It seems reasonable to consider only those systems which respect the complexified action.

Example 46. Let $\Gamma := \mathbb{R}^*$ act on $\mathbb{R}^2$ by $(r,(x,y)) \mapsto (r^2 x, r^{-2} y)$. Again $\rho(x,y) = xy$ is a Hilbert map. The orbits $\Gamma(1,1) = \{(x,y) \mid xy = 1 \wedge x > 0\}$ and $\Gamma(-1,-1) = \{(x,y) \mid xy = 1 \wedge x < 0\}$ have the same image.

The complexification of $\mathbb{R}^*$ is $\Gamma_{\mathbb{C}} := \mathbb{C}^*$. Both orbits $\Gamma(1,1)$ and $\Gamma(-1,-1)$ are subsets of $\Gamma_{\mathbb{C}}(1,1) = \{(x,y) \mid xy = 1\}$.

An important difference to the compact case is that there are representations of non-compact groups which do not have a non-constant invariant polynomial. In this case eigenvalues belonging to such a representation may not be found by the reduced system.

Example 47. Consider the action of $\mathbb{R}^*$ on $\mathbb{R}^2$ by $(r,(x,y)) \mapsto (rx, r^{-1}y)$. The algebra of invariant polynomials is generated by $\rho(x,y) = xy$. The space of closed orbits is $\mathbb{R}$.

The equivariant linear map $P(x,y) = (\lambda x, \mu y)$ is projected to $Q(z) = \lambda\mu z$. Although it is true that the eigenvalues of $DQ(0)$ are products of the eigenvalues of $DP(0)$ (Theorem 6), the stability of P is not determined by $DQ(0)$, i.e., Corollary 8 no longer holds.

Example 47 shows that the reduced map does not carry enough information about the eigenvalues of the original system.

At points with compact isotropy subgroup this problem does not arise, because the algebraic quotient has the structure of the orbit space of the compact isotropy group near such points.

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