Dimension Theory of Smooth Dynamical Systems

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1 Introduction

One of the basic properties of dynamical systems is that local instability of trajectories gives rise to a global "chaotic" behavior. This local instability can be described as some kind of hyperbolicity. Smooth Ergodic Theory investigates the metric and stochastic properties of measures invariant under differentiable mappings or flows on manifolds. The consideration of invariant measures allows to "tame" the "chaotic" behavior from a probabilistic point of view. This transition from differentiable structures to measurable structures and vice versa makes this field fascinating and paves the way to applications far beyond this field. Due to its generality the methods and results of Smooth Ergodic Theory entered areas as Riemannian Geometry, Number Theory, Statistical Physics, Partial Differential Equations or Numerical Simulations.

In order to study the properties of an invariant measure one considers different kind of characteristics of it. This characteristical quantities entered from both the theoretical and experimental investigations. Among them the most important ones are *Lyapunov exponents* – a metric quantity measuring the local instability, *entropy* – a stochastic quantity measuring the complexity of the system, and fractal dimesions – quantities having both metric and stochastic aspects. While the first two characteristics have a commonly aggreed definition the fractal dimensions collect a variety of similar but different notions according to the usefulness in applications.

In the late 70's – beginning 80's the attention of many physicists and applied mathematicians has turned to the study of dimension of strange attractors. The dimension was used to characterize the (finite) number of independent modes needed to describe the infinite-dimensional system. Several conjectures were put forward in attempts to tie together the dimension of an attractor and its other characteristics. This study has become an important breakthrough in understanding the structure of systems of evolution type.

The seminal paper [15] by Eckmann and Ruelle was one of the most serious attempts to comprehend this activity and outline a rigorous mathematical foundation for it. They discussed various concepts of dimension and ruled out the importance of the so-called pointwise dimension. The discussion in [15] can be summarized in the following conjecture (*Eckmann–Ruelle–Conjecture*):

Let M be a compact smooth Riemannian manifold without boundary, $f\colon M \to M$ a C^2–diffeomorphism, and μ an f–invariant Borel probability measure on M. Then if μ is hyperbolic, *i.e. all the Lyapunov exponents of μ are non–zero, the following limit exists*

$$d(x) := \lim_{r\to} \frac{\log \mu(B(x,r))}{\log r}$$

for a.e. $x \in M$.

The existence of this limit implies that virtually **all** the characteristics of dimension type of the measure coincide and it makes sense to talk about the fractal dimension of the measure. Moreover, the interplay of the metric and stochastic properties of the measure becomes a more clear and rigid structure.

In [5] Barreira, Pesin and Schmeling could proof the above conjecture in full generality, even the assumption that f has to be C^2 could be weakened to f belonging to the class $C^{1+\alpha}$, $\alpha > 0$. The proof of this conjecture consists of two major parts. First the work of Ledrappier and Young [25] was generalized to the class $C^{1+\alpha}$ in the case of hyperbolic measures and second it was established that arbitrary hyperbolic measures admit a very special local property: they exhibit "almost" the so-called local product structure.

Relying on this result Schmeling and Troubetzkoy [49] established a more general scaling law of hyperbolic measures for 3–dimensional systems. Instead of the ball $B(x,r)$ one can consider rectangles aligned along the invariant directions with exponentially increasing distortion. The existence of a scaling along these rectangles recovers the existence of the pointwise dimension as well as the Brin–Katok formula for the local entropies and gives a new insight in the connection in between entropy and dimension. F. Ledrappier found a similar scaling law for Takens balls (i.e. rectangles with special relations between their side–lengths) in any dimension. These general scaling laws allowed to proof the existence of the pointwise dimension for non–invertible systems with singularities.

2 Smooth Dynamical Systems

In order to develop a general theory we try to keep the assumptions on the systems considered as weak as possible. We are interested in the connection of topological (geometric), measure theoretic and other properties of these systems. In particular we want to describe the fascinating interaction of determinism and randomness. Let M be a smooth compact connected Riemannian manifold with distance d and $f\colon M \to M$ a $C^{1+\alpha}$ diffeomorphism.

A point $x \in M$ is called *Lyapunov regular* if there exists numbers

$$\lambda_1(x) < \cdots < \lambda_{k(x)}(x)$$

called *Lyapunov exponents* and a decomposition $T_xM = \bigoplus_{i=1}^{k(x)} E^i(x)$ of the tangent space, where $E^i(x)$ is the vector space

$$E^i(x) = \left\{ v \in T_xM \setminus \{0\} : \lim_{n\to\pm\infty} \frac{1}{n} \log \|d_x f^n v\| = \lambda_i(x) \right\} \cup \{0\}$$

such that

$$\lim_{n\to\pm\infty} \frac{1}{n} \log |\det d_x f^n| = \sum_{i=1}^{k(x)} \lambda_i(x) \dim E^i(x).$$

Note that $\lambda_i(f(x)) = \lambda_i(x)$ and $d_x f E^i(x) = E^i(f(x))$ for each i.

We denote by $\Lambda \subset M$ the set of all Lyapunov regular points. Below we describe some notions and results on non-uniformly hyperbolic dynamical systems. For references see [27,28,25,21].

Set $s(x) = \max\{i : \lambda_i(x) < 0\}$ and $u(x) = \min\{i : \lambda_i(x) > 0\}$. Fix $x \in \Lambda$ and $r(x) > 0$. For every $i = 1, \ldots, s(x)$ one defines the *i-th stable leaf* at x by

$$W^i(x) = \left\{ y \in B(x, r(x)) : \limsup_{n\to+\infty} \frac{1}{n} \log d(f^n(x), f^n(y)) < \lambda_i(x) \right\},$$

and for every $i = u(x), \ldots, k(x)$ the *i-th unstable leaf* at x by

$$W^i(x) = \left\{ y \in B(x, r(x)) : \limsup_{n\to-\infty} \frac{1}{n} \log d(f^n(x), f^n(y)) > \lambda_i(x) \right\}.$$

Clearly, $W^i(x) \subset W^{i+1}(x)$ if $i < s(x)$ and $W^i(x) \supset W^{i+1}(x)$ if $i \geq u(x)$. We write $W^s(x) = W^{s(x)}(x)$ and $W^u(x) = W^{u(x)}(x)$ and call them, respectively, the *stable* and *unstable manifolds* at x. One can prove that if $r(x)$ is sufficiently small than $W^i(x)$ is a $C^{1+\alpha}$ immersed submanifold.

One of the major questions in the theory is the degree of regularity of these foliations. The importance of the regularity started with Hopf's method of proving the ergodicity of the geodesic flow on a surface of negative curvature. He used the foliations to define new coordinates. The smoothness of the coordinates obviously depends on the regularity of the foliations. Another important application found its way in a series of rigidity results. As one example Hurder and Katok (see [19]) proved that an Anosov diffeomorphism of the 2–dimensional torus is C^∞– conjugated to a linear one under quite mild regularity assumption on the invariant foliations. By a number of authors (see f.i. [46,18]) it was realized that in general the stable or unstable foliations are only Hölder continuous. Optimal Hölder exponents were found and it was proved that these exponents are optimal on a set of second category in the space of all diffeomorphisms of a manifold ([40]). Hasselblatt and Wilkinson even found an open set of symplectic Anosov diffeomorphisms were the least regularity is obtained on a set of full measure on the torus. Nevertheless the situation is different if we consider strong stable foliations within a weak stable leaf.

Let us fix $c > 0$, $x \in \Lambda_c$, where Λ_c is a subset of Λ consisting of points having the same collection of Lyapunov exponents and all i–th stable leaves of inner diameter at least c, and $y' \in \Lambda_c \cap B^{i+1}(x, c)$. For each $i < s$, consider two local smooth manifolds Σ_x and $\Sigma_{y'}$ in $W^{i+1}(x)$, containing x and y', respectively and transversal to $W^i(z)$ for all $z \in \Lambda_c \cap B^{i+1}(x, c)$. The holonomy map

$$\Pi_i = \Pi_i(\Sigma_x, \Sigma_{y'}) \colon \Sigma_x \cap \Lambda_c \cap B^{i+1}(x, c) \to \Sigma_{y'}$$

is defined by

$$\Pi_i(x') = W^i(x') \cap \Sigma_{y'}$$

with $x' \in \Sigma_x$. This map is well-defined if c is sufficiently small.

In [5] we proved the following theorem.

Theorem 1. *Let f be a $C^{1+\alpha}$ diffeomorphism. For $x \in \Lambda_c$, and $y' \in \Lambda_c \cap B^i(x, c)$ the holonomy map $\Pi_i(\Sigma_x, \Sigma_{y'})$ is Lipschitz continuous.*

This theorem was obtained by a different method by Ledrappier and Young [25] for the case of a C^2–diffeomorphism.

The natural question that arises is how large is the set we were talking about – i.e. how many regular points exist? Here the connection of geometry and measure theory comes into play. Namely, the famous theorem of Oseledec states that any invariant measure is concentrated on the set of regular points. Moreover, if we assume that the measure is ergodic it has to be concentrated on a set where the Lyapunov exponents are constant. We want to emphazise that Oseledec theorem is no longer true if we allow the system to have singularities or to be non–invertible.

For the following theory we need an additional assumption. We call a point $x \in M$ *non-uniformly hyperbolic* iff $E^0 = \{0\}$. An ergodic measure is called hyperbolic iff it is concentrated on the non–uniformly hyperbolic points.

Now we can ask what are the **interesting characteristics** of a smooth dynamical system? We encountered already the Lyapunov exponents which describe the system from the point of view of the underlying Riemannian metric. Another classical characteristic is the *entropy* which measures the inner complexity of the system. In the recent past another characteristic came into the center of interest. It is the *Hausdorff dimension* which was found to connect the previous quantities and to contain the combined information. The main role was played by the local version of the dimension, the pointwise dimension of a measure:

$$d_\mu(x) = \lim_{r \to 0} \frac{\log \mu B(x, r)}{\log r}.$$

One of the main problems is the existence of the above limit. As in the theory of entropies (the theorem of Shannon–McMillan–Breiman) the existence ensures a coincidence of different concepts of the dimension of a measure and also the connection to the other characteristics.

In an extremely important and also extremely difficult paper ([25]) Ledrappier and Young defined entropies and pointwise dimensions $d^i_\mu(x)$ w.r.t. conditional measures sitting on the i–th (un–)stable leaves. They were able to prove the a.e. existence of these quantities and stated the following formula

$$h_\mu = -\sum_{j\le s} \chi_j (d^j_\mu - d^{j-1}_\mu)$$

This put us in the situation where we know that Lyapunov exponents, entropies and stable and unstable dimensions exist a.e. and that there is a connection between the conditional quantities. There is a connection in between the unconditioned quantities if the pointwise dimension exist. So the question is when does the pointwise dimension of a measure exist? For an hyperbolic measure (i.e. no indifferent directions) a natural guess is that the pointwise dimension exists a.e. and equals the sum of the stable and unstable dimension. Unfortunately, a simple Fubini type argument does not work since the foliations are not regular enough in general.

In [5] we prove that every hyperbolic measure invariant under a $C^{1+\alpha}$ diffeomorphism of a smooth Riemannian manifold possesses asymptotically "almost" local product structure, i.e., its density can be approximated by the product of the densities on stable and unstable manifolds up to small exponentials. This has not been known even for measures supported on locally maximal hyperbolic sets.

Using this property of hyperbolic measures we were able to prove the long-standing Eckmann–Ruelle conjecture in dimension theory of smooth dynamical systems: the pointwise dimension of every hyperbolic measure invariant under a $C^{1+\alpha}$ diffeomorphism exists almost everywhere. This implies the crucial fact that virtually all the characteristics of dimension type of the measure (including the Hausdorff dimension, box dimension, and information dimension) coincide. This provides the rigorous mathematical justification of the concept of fractal dimension for hyperbolic measures.

The problem of the existence of the pointwise dimension has a long history. In [55], Young obtained a positive answer for a hyperbolic measure μ invariant under a $C^{1+\alpha}$ surface diffeomorphism f. Moreover, she showed that in this case for almost every point x

$$\underline{d}_\mu(x) = \overline{d}_\mu(x) = h_\mu(f)\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right),$$

where $h_\mu(f)$ is the metric entropy of f and $\lambda_1 > 0 > \lambda_2$ are the Lyapunov exponents of μ.

In [23], Ledrappier established the existence of the pointwise dimension for arbitrary *SRB-measures* (called so after Sinai, Ruelle, and Bowen). In [35], Pesin and Yue extended his approach and proved the existence for hyperbolic measures satisfying the so-called *semi-local product structure* (this class includes, for example, *Gibbs measures* on locally maximal hyperbolic sets).

Our main result in [5] is the following theorem.

Theorem 2. *Let f be a $C^{1+\alpha}$ diffeomorphism on a smooth Riemannian manifold M without boundary, and μ an f-invariant compactly supported ergodic Borel probability measure. If μ is hyperbolic then the following properties hold:*

- *for every $\delta > 0$ there exist a set $\Lambda \subset M$ with $\mu(\Lambda) > 1 - \delta$ and a constant $\kappa \geq 1$ such that for every $x \in \Lambda$ and every sufficiently small r (depending on x), we have*

$$r^{\delta}\mu_x^s(B^s(x,\frac{r}{\kappa}))\mu_x^u(B^u(x,\frac{r}{\kappa})) \leq \mu(B(x,r)) \leq r^{-\delta}\mu_x^s(B^s(x,\kappa r))\mu_x^u(B^u(x,\kappa r));$$

- *μ is exact dimensional and its pointwise dimension is equal to the sum of the stable and unstable pointwise dimensions, i.e.,*

$$\underline{d} = \overline{d} = d^s + d^u.$$

Let us also point out that neither of the assumptions of the Main Theorem can be omitted. Ledrappier and Misiurewicz [24] constructed an example of a smooth map of a circle preserving an ergodic measure with zero Lyapunov exponent which is not exact dimensional. In [33], Pesin and Weiss presented an example of a Hölder homeomorphism with Hölder constant arbitrarily close to 1 whose measure of maximal entropy is not exact dimensional.

Statement 1 of the Main Theorem establishes a new and non-trivial property of an arbitrary hyperbolic measure. Loosely speaking it means that every hyperbolic invariant measure possesses asymptotically "almost" local product structure. This statement has not been known even for measures supported on (uniformly) hyperbolic locally maximal invariant sets.

3 Endomorphisms with singularities

The previous sections indicated that the dimension theory of diffeomorphisms exhibiting an hyperbolic measure is well understood. But how realistic are these systems? Often PDE's generate only semi–flows – i.e. the system is a priori non–invertible. Moreover, after reducing the system to a system of ODE's the investigation of Poincare sections induces singularities. So we are far away from the settings we studied so far. The very first question arises how to find physical meaningful measures (those measures are called *SBR–measures*)?

In this section we consider maps of a manifold which may have singularities and are non–invertible. Let $M = \bigcup K_i \cup N$ where N is the finite union of smooth lower-dimensional submanifolds. The restriction $f|_{K_i}$ is a

C^2–diffeomorphism onto its image and has a smooth extension to the boundary of K_i.

We lift the map f to a map $\hat{f}$ which is invertible. Namely, we set for $I = [0,1]$ and $\tau > 0$ sufficiently small

$$\hat{M} := M \times I, \quad \hat{K}_i := K_i \times I, \quad \hat{x} := (x, \omega) \in \hat{M};$$
$$\hat{f}(x, \omega) := \left(fx, \tau\omega + \frac{i-1}{r} \right), \text{ for } x \in K_i, \ \omega \in I, \ i = 1, ..., r.$$

Let $\pi : \hat{M} \to M$ be the projection of $\hat{M}$ onto the first coordinate. Then $f\pi = \pi\hat{f}$ and $\hat{f}$ is invertible. We use this solenoidal construction rather than the inverse limit space of the map f (see [36]) since the inverse limit space is not a manifold and we cannot apply the theory of diffeomorphisms with singularities to it.

Definition 3. Let $\hat{\Lambda}$ be the set of regular points of $\hat{f}$. We say that an invariant Borel probability measure m is *regular hyperbolic* if its lift is such that $\hat{m}(\hat{\Lambda}) = 1$ and all of its Lyapunov exponents are non-zero– i.e. $s(\hat{z}) + 1 = u(\hat{z})$ a.e. We also call $\hat{m}$ regular hyperbolic.

One of the main examples of a regular hyperbolic measure is the **SBR-measure**. It has been shown to exist for a large class of hyperbolic endomorphisms [49].

In [50] we proved the following generalization of the main result in the previous section.

Theorem 4. *Suppose f is an endomorphism with singularities described above and that m is a regular hyperbolic measure for f. Then the limit*

$$d_m(x) := \lim_{\varepsilon \to 0} \frac{\log mB(x, \varepsilon)}{\log \varepsilon} = d^s_{\hat{M}}(\hat{x}) + d^u_{\hat{M}}(\hat{x}) - d^1_{\hat{M}}(\hat{x})$$

exists m–almost everywhere. If m is additionally ergodic then $d_m(x)$ is constant a.e.

4 Applications in 2 dimensions

In this section we will indicate some improvement of the previous results in dimension 2.

If m is an SBR–measure of an endomorphism with singularities on a surface then its *Lyapunov dimension* is defined as

$$\dim_L m = 1 - \frac{\lambda^u}{\lambda^s}.$$

We note that an SBR–measure on a surface always has exactly one negative and one positive exponent.

We say that a system is *a.s. invertible w.r.t. m* if there is a set of full measure on which the restriction of f is invertible. This means that up to measure 0 a system has unique forward and backward trajectories on the attractor and that the semi–flow is actually a flow restricted to this set of full measure.

In [49] a class of piecewise smooth endomorphisms with hyperbolic attractors on surfaces was introduced. We developed the ergodic theory of such systems. In particular we proved the existence of the SBR-measure and gave a criterion which connects the almost sure invertibility of the map on the attractor and the dimension of the set. This result can be stated in the following way.

Theorem 5. *The system is almost surely invertible (w.r.t. the SBR-measure) on the attractor if and only if the dimension of the SBR-measure equals its Lyapunov dimension.*

In [43] we applied the general theory from [49] to a specific example. The reason is that we want to illustrate the helpfulness of the above mentioned criterion. Our example consists of a three-parameter family of piecewise smooth maps. Our main attention is paid to the question for which parameters the a priori non-invertible system is invertible on a set of full SBR–measure. From the first point of view this example seems rather special. But it contains almost all difficulties of the general case and it is the most natural one for further investigations.

We consider the three-parameter family of the Belykh map.Let us consider the square $Q = [-1,1] \times [-1,1] \in \mathbb{R}^2$ and the map $f : Q \to Q$ defined by

$$f(x,y) = \begin{cases} (\lambda x_1 + 1 - \lambda, \gamma x_2 + 1 - \gamma) & x_2 > kx_1 \\ (\lambda x_1 + (\lambda - 1), \gamma x_2 + (\gamma - 1)) & x_2 < kx_1 \end{cases} \tag{1}$$

with $-1 < k < 1, 1 < \gamma \leq \frac{2}{|k|+1}, 0 < \lambda \leq 1$.

The Belykh map is natural for non-invertibility and features a lot of properties of the maps in the class introduced in [49].For the parameter range where the map is invertible it was first introduced by V.P. Belykh in [9] as a simple model for phase synchronization arising as a model for the Poincare map of a system of differential equations. The ergodic properties of this system in the parameter range of invertibility were investigated by Ya.B. Pesin in [29] and this system served as an example of the investigations of E.A. Sataev in [39] and later in [1]. J. Alexander and J. Yorke [2] considered a special case of this system for parameters where it is completely non-invertible. They called it the fat baker's transformation. We generalized the above examples (including the original baker's transformation) in a natural way and obtain a family of maps which increase the amount of non-invertibility as the parameters change. We believe that this family reflects most of the properties of generic parameter families of hyperbolic piecewise smooth maps.

Using the theory of [49] we can conclude the existence of an SBR–measure and investigate its properties. In particular, we are interested in the question how the Hausdorff dimension of the measure depends on the parameters. By the above criterion this is connected to the almost sure invertibility of the map.

Theorem 6.

– *If* $\lambda \cdot \gamma^2 < 1$ *then* $f_{\lambda,\gamma,k}$ *is fully invertible on the attractor for Lebesgue a.e.* (λ, γ, k). *Hence,*

$$\dim_H m = 1 - \frac{\log \gamma}{\log \lambda}$$

– *If* $\lambda \cdot \gamma < 1$ *then* $f_{\lambda,\gamma,k}$ *is a.s. invertible on the attractor for Lebesgue a.e.* (λ, γ, k). *Hence,*

$$\dim_H m = 1 - \frac{\log \gamma}{\log \lambda}$$

– *If* $\lambda \cdot \gamma > 1$ *then* $f_{\lambda,\gamma,k}$ *for Lebesgue a.e.* (λ, γ, k)

$$\dim_H m = 2 < \dim_L m$$

Remark 7. The above theorem shows that the Lyapunov dimension equals the Hausdorff dimension of the SBR-measure for almost all Belykh maps with $\lambda \cdot \gamma < 1$. This is the well-known Kaplan-Yorke conjecture for the special case of the Belykh attractor.

5 Irregular points

In the previous sections we investigated the behavior of the system on the sets of regular points. Our main motivation was that in many situations these points carry full invariant measure. This study gives important information about the observable properties of the dynamical system, and "typical" points with respect to different measures give complementary information.

The set of "non-typical" points, i.e., the set of points that is "typical" with respect to no measure, has rarely been considered in the literature. In [3] we show that, surprisingly, in several situations central in the theory of dynamical systems this set contains complete information about some observable properties. Namely, the set of "non-typical" points carries **full** topological entropy and **full** Hausdorff dimension. In this sense the irregular points occupy a substantial part of the phase space.

5.1 Repellers and Horseshoes

Let $f\colon M \to M$ be a C^1 map of a smooth manifold, and J an f-invariant compact subset of M. We say that f is **expanding** on J and that J is a **repeller** of f if there are constants $C > 0$ and $\beta > 1$ such that $\|d_x f^n u\| \geq C\beta^n \|u\|$ for all $x \in J$, $u \in T_x M$, and $n \geq 1$.

It is well known that repellers admit Markov partitions of arbitrarily small diameter. Each Markov partition has associated a one-sided subshift of finite type $\sigma|\Sigma$, and a **coding map** $\chi\colon \Sigma \to J$ for the repeller, which is Hölder continuous, onto, and satisfies $f \circ \chi = \chi \circ \sigma$ and $\sup\{\mathrm{card}(\chi^{-1}x)\colon x \in J\} < \infty$ (see, for example, [30] for details).

A differentiable map $f\colon M \to M$ is called **conformal** on a set J if $d_x f$ is a multiple of an isometry at every point $x \in J$. Well-known examples of conformal expanding maps include one-dimensional Markov maps and holomorphic maps. We write $a(x) = \|d_x f\|$ for each $x \in M$. For a repeller J of a conformal $C^{1+\varepsilon}$ expanding map f, the equilibrium measure m_D of $-\dim_H J \cdot \log a$ on J is called the **measure of maximal dimension** (m_D is the unique f-invariant measure μ such that $\dim_H \mu = \dim_H J$; see, for example, [30] for details). We denote by m_E the **measure of maximal entropy**, i.e., the equilibrium measure of 0.

A uniformly hyperbolic basic set (*horseshoe*) of a surface diffeomorphism can be coded by a two–sided subshift of finite type in a similar manner.

5.2 Irregular sets for Birkhoff averages

Let $C(\Sigma)$ be the space of continuous functions on Σ. For each function $g \in C(\Sigma)$, we define the **irregular set for the Birkhoff averages of** g by

$$\mathcal{B}(g) = \left\{ x \in \Sigma\colon \lim_{n\to\infty} \frac{1}{n} S_n g(x) \text{ does not exist} \right\},$$

where

$$S_n g(x) = \sum_{k=0}^{n} g(\sigma^k x) \tag{2}$$

for each $x \in \Sigma$ and $n \in \mathbb{N}$. By the Birkhoff Ergodic Theorem, $\mu(\mathcal{B}(g)) = 0$ for every σ-invariant measure μ.

The following statement shows that the zero measure set $\mathcal{B}(g)$ is "observable"; namely, $\mathcal{B}(g)$ carries full topological entropy for a large class of functions g.

Theorem 8. *Let $\sigma|\Sigma$ be a topologically mixing subshift of finite type, and $g_1, \dots, g_k$ Hölder continuous functions on Σ. Then the following properties are equivalent:*

1. *the functions $g_1, \dots, g_k$ are non-cohomologous to 0;*

2. $\mathcal{B}(g_1) \cap \cdots \cap \mathcal{B}(g_k)$ *is non-empty;*
3. $\mathcal{B}(g_1) \cap \cdots \cap \mathcal{B}(g_k)$ *is a proper dense subset;*
4. $h(\sigma|\mathcal{B}(g_1) \cap \cdots \cap \mathcal{B}(g_k)) > 0$*;*
5. $h(\sigma|\mathcal{B}(g_1) \cap \cdots \cap \mathcal{B}(g_k)) = h(\sigma)$*.*

Remark 9. If $\sigma|\Sigma$ is uniquely ergodic, then the set $\mathcal{B}$ is empty.

5.3 Irregular sets for local entropies

For each probability measure μ on Σ, we define the **irregular set for the local entropies of** μ by

$$\mathcal{H}(\mu) = \left\{ x \in \Sigma \colon \lim_{n\to\infty} -\frac{\log \mu(C_n(x))}{n} \text{ does not exist} \right\},$$

where $C_n(x)$ denotes the cylinder of length n which contains the point $x \in \Sigma$. The set $\mathcal{H}(\mu)$ is σ-invariant but may not be compact.

Remark 10. 1. If μ is σ-invariant, then $\mu(\mathcal{H}(\mu)) = 0$ (using the Shannon–McMillan–Breiman Theorem).
2. If μ is a Gibbs measure, then there is a cohomology class of functions in $C(\Sigma)$ such that $\mathcal{H}(\mu) = \mathcal{B}(g)$ if and only if g belongs to this cohomology class.
3. Let μ be a σ-invariant measure of maximal entropy. If μ is a Gibbs measure, then $\mathcal{H}(\mu)$ is empty. For example, if $\sigma|\Sigma$ is a topologically mixing subshift with the specification property (in particular, if $\sigma|\Sigma$ is a subshift of finite type or a sofic subshift), then any σ-invariant measure of maximal entropy is a Gibbs measure.

We define the **irregular set for the Lyapunov exponents of** f by

$$\mathcal{L}_f = \left\{ x \in J \colon \lim_{n\to\infty} \frac{1}{n} \log\|d_x f^n\| \text{ does not exist} \right\},$$

and for each probability measure μ on J, the **irregular set for the pointwise dimensions of** μ by

$$\mathcal{D}(\mu) = \left\{ x \in J \colon \lim_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} \text{ does not exist} \right\},$$

where $B(x,r) \subset J$ is the ball of radius r centered at x.

For a repeller J of a topologically mixing expanding map, if any of the invariant sets $\mathcal{B}_f$, $\mathcal{H}_f(\mu)$, $\mathcal{L}_f$, and $\mathcal{D}(\mu)$ is non-empty, then it is dense in J.

By Kingman's Subadditive Ergodic Theorem, we have $\mu(\mathcal{L}_f) = 0$ for any f-invariant probability measure μ on J. In [48] we proved that if μ is a measure invariant under an expanding map (not necessarily conformal), then $\mu(\mathcal{D}(\mu)) = 0$.

Theorem 11. *If J is a repeller of a topologically mixing $C^{1+\varepsilon}$ expanding map f, for some $\varepsilon > 0$, and f is conformal on J, then the following properties hold:*

1. $h(f|\mathcal{B}_f) = h(f|J)$ *and* $\dim_H \mathcal{B}_f = \dim_H J$;
2. $m_D \neq m_E$ *if and only if* $h(f|\mathcal{L}_f) = h(f|J)$ *and* $\dim_H \mathcal{L}_f = \dim_H J$.

If in addition $\mu \in G(f|J)$ *then*

3. $\mu \neq m_D$ *if and only if* $h(f|\mathcal{D}(\mu)) = h(f|J)$ *and* $\dim_H \mathcal{D}(\mu) = \dim_H J$;
4. $\mu \neq m_E$ *if and only if* $h(f|\mathcal{H}_f(\mu)) = h(f|J)$ *and* $\dim_H \mathcal{H}_f(\mu) = \dim_H J$;
5. *the three measures* μ, m_D, *and* m_E *are distinct if and only if*
$$h(f|\mathcal{D}(\mu) \cap \mathcal{H}_f(\mu) \cap \mathcal{L}_f) = h(f|J)$$
and
$$\dim_H(\mathcal{D}(\mu) \cap \mathcal{H}_f(\mu) \cap \mathcal{L}_f) = \dim_H J.$$

It was established by Shereshevsky in [51] that $\dim_H \mathcal{D}(\mu) > 0$, and $\overline{\mathcal{D}(\mu)} \supset \Lambda$ for a generic C^2 surface diffeomorphism possessing a locally maximal hyperbolic set Λ, and a generic Hölder continuous potential, with respect to the C^0 topology, with Gibbs measure μ. In [3] we were able to improve this result.

Theorem 12. *For a surface diffeomorphism f in a C^2 open dense set, possessing a compact locally maximal hyperbolic set Λ, then*

$$\dim_H(\mathcal{D}(\mu) \cap \mathcal{H}_f(\mu) \cap (\mathcal{L}_f \cup \mathcal{L}_{f^{-1}})) > \sup_{\nu} \dim_H \mathcal{G}(\nu).$$

for any Gibbs measure μ and where $G(\nu)$ denotes the set of generic points for the ergodic measure ν.

6 Multifractal analysis

The multifractal analysis, i.e., the analysis of invariant sets and measures with multifractal structure, has been recently developed as a powerful tool for numerical study of dynamical systems. These spectra capture information about various dimensions associated with the dynamics. Among them are the well-known Hausdorff dimension, correlation dimension, and information dimension of invariant measures.

Another example of multifractal spectra is entropy spectra introduced in [6]. They provide an integrated information on the distribution of topological entropy associated with local entropies.

In [6,7] it is demonstrated that multifractal spectra can be used in a sense to "restore" the dynamics – the phenomenon that we call *multifractal rigidity*.

The multifractal analysis is essentially measuring the "size" (in the sense of Hausdorff dimension or topological entropy) of the level sets of special discontinuous measurable functions (such as the local entropy, the pointwise dimension, etc.) on geometrically complicated objects (supports of invariant measures). This is expressed in a function f which is called the *spectrum*. In this sense the multifractal analysis can be considered as a non-smooth Morse theory. Our main concern is the further investigation of the dimension spectrum (i.e. the Hausdorff dimension of the level sets of the pointwise dimension) and the entropy spectrum (i.e. the topological entropy of the level sets of local entropies).

This analysis was investigated in several different situations and there is a huge amount of literature on this subject (see [30] for references and more details). One of the main results in this theory is that there is an interval (α_1, α_2) on which the spectrum $f(\alpha)$ (for definition see below) is analytic, convex, and can be continuously extended to the boundary provided it is not a point spectrum.

6.1 A general concept of multifractality

In this section we follow an outline of [6]. The invariant sets of a dynamical system may have a complicated geometric structure. In general, these sets are not self-similar, but can often be decomposed into subsets each possessing some scaling symmetry. This decomposition is called a *multifractal decomposition* and is an essential part of the multifractal analysis of dynamical systems.

Dimension spectra are the historically first examples of more general multifractal spectra which we introduce in [6].

One of the main points is to demonstrate that multifractal spectra can be used in a sense to "restore" the dynamics – the phenomenon that we call the *multifractal rigidity*. There are two main problems related to multifractal rigidity. First, given a dynamical system of hyperbolic type there exist *finitely* many *independent* multifractal spectra which uniquely identify the main *macro-characteristics* of the system (such as its invariant measure, geometric structure of its invariant sets, their dimensions, etc.). These spectra can be viewed as a special type of *degrees of freedom*, called *multifractal degrees of freedom*, and can be effectively used in the numerical study of dynamical systems.

In particular, for subshifts of finite type and some conformal expanding maps, the dimension spectrum alone is sufficient to determine all other multifractal spectra, and thus, these systems have one multifractal degree of freedom.

Another problem is inspired by an attempt to produce a "physically meaningful" classification of dynamical systems which takes care of various aspects of the dynamics (chaoticity, instability, geometry, etc.) simultaneously.

In the theory of dynamical systems there are various types of classifications. The most prominent ones seem to be *topological classification* (up to homeomorphisms) and *measure-theoretic classification* (up to measure preserving automorphisms). From a physical point of view, these classifications trace separate "independent" characteristics of the dynamics. We suggest a new type of classification which is based upon multifractal spectra and combines features of each of the above classifications, in what we call the *multifractal classification*. The new classification has a strong physical content and identifies two systems up to a change of variables. From a mathematical point of view, we establish the smooth equivalence of two dynamical systems which are *a priori* only topologically equivalent and have the same multifractal degrees of freedom. The multifractal classification is much more rigid than the topological and measure-theoretic classifications. Besides the smooth equivalence of the two dynamical systems, it establishes the coincidence of their dimension characteristics as well as the correspondence between their invariant measures.

We begin with the general concept of multifractal spectrum.

Let X be a set and let $g\colon X \to [-\infty, +\infty]$ be a function. The *level sets* of g

$$K_\alpha^g = \{x \in X : g(x) = \alpha\}, \quad -\infty \le \alpha \le +\infty$$

are disjoint and produce a *multifractal decomposition* of X, that is,

$$X = \bigcup_{-\infty \le \alpha \le +\infty} K_\alpha^g. \tag{3}$$

Let now G be a set function, i.e., a real function that is defined on subsets of X. Assume that $G(Z_1) \le G(Z_2)$ if $Z_1 \subset Z_2$. We define the function $\mathcal{F}\colon [-\infty, +\infty] \to \mathbb{R}$ by

$$\mathcal{F}(\alpha) = G(K_\alpha^g).$$

We call $\mathcal{F}$ the *multifractal spectrum* specified by the pair of functions (g, G), or the (g, G)-multifractal spectrum. The function $\mathcal{F}$ captures an important information about the structure of the set X generated by the function g.

It often happens that the function g is defined only on a subset $Y \subset X$. In this case the decomposition (3) should be replaced by

$$X = \bigcup_{-\infty \le \alpha \le +\infty} K_\alpha^g \cup (X \setminus Y).$$

We still call this decomposition of X a *multifractal decomposition*.

6.2 Examples of multifractal spectra

We illustrate the general concept of multifractal spectra by studying several explicit spectra.

6.3 Dimension and entropy spectra

Let X be a complete separable metric space and let $f\colon X \to X$ be a continuous map. We begin with the choice of the set function G. There are two "natural" set functions on X. The first one is generated by the metric structure on X. Namely, given a subset $Z \subset X$, we set

$$G_D(Z) = \dim_H Z, \tag{4}$$

where $\dim_H Z$ is the Hausdorff dimension of Z.

The second function is generated by the dynamical system f acting on X and the metric on X. Namely,

$$G_E(Z) = h(f|_Z), \tag{5}$$

where $h(f|_Z)$ is the topological entropy of f on Z (notice that Z need not be compact nor f-invariant and one can use the definitions in [10,45]). We call the multifractal spectra generated by the function G_D *dimension spectra*, and the multifractal spectra generated by the function G_E *entropy spectra*.

We now describe some "natural" choices for the function g.

6.4 Multifractal spectra for pointwise dimensions

Let μ be a Borel finite measure on X. Consider the subset $Y \subset X$ consisting of all points $x \in X$ for which the pointwise dimension $d_\mu(x)$ of the measure μ exists. We define the function g_D on Y by

$$g_D(x) = d_\mu(x).$$

We note that the corresponding multifractal decomposition consists of the sets

$$K_\alpha^{g_D} = \{x : d_\mu(x) = \alpha\}.$$

We obtain two multifractal spectra $\mathcal{D}_D = \mathcal{D}_D^{(\mu)}$ and $\mathcal{D}_E = \mathcal{D}_E^{(\mu)}$ specified by the pairs of functions (g_D, G_D) and (g_D, G_E), respectively, where the set functions G_D and G_E are given by (4) and (5). We call them *multifractal spectra for (pointwise) dimensions.*

Let us remark that the spectrum $\mathcal{D}_D$ is known in the literature as the *dimension spectrum* or $f_\mu(\alpha)$*-spectrum for dimensions.* The concept of a multifractal analysis was suggested by a group of physicists in [17] (see [30] for more references and details).

6.5 Multifractal spectra for local entropies

Let X be a complete separable metric space and let $f\colon X \to X$ be a continuous map preserving a Borel probability measure μ. Consider the set $Y \subset X$

consisting of all points $x \in X$ for which the local entropy $h_\mu(x)$ exists. We define the function g_E on Y by

$$g_E(x) = h_\mu(x).$$

We note that the corresponding multifractal decomposition consists of the sets

$$K_\alpha^{g_E} = \{x : h_\mu(x) = \alpha\}.$$

We obtain two multifractal spectra $\mathcal{E}_D = \mathcal{E}_D^{(\mu)}$ and $\mathcal{E}_E = \mathcal{E}_E^{(\mu)}$ specified by the pairs of functions (g_E, G_D) and (g_E, G_E), respectively, where the set functions are given by (4) and (5). We call them *multifractal spectra for (local) entropies.*

We want to emphazise that the above definitions are dependent on the choice of a partition. But in the situations we consider later there are canonical partitions on which the spectra do not depend.

6.6 Multifractal spectra for Lyapunov exponents

Let X be a differentiable manifold and let $f: X \to X$ be a C^1 map. Consider the subset $Y \subset X$ of all points $x \in X$ for which the Lyapunov exponent $\lambda_u(x)$ exists. We define the function g_L on Y by

$$g_L(x) = \lambda(x).$$

We note that the corresponding multifractal decomposition consists of the sets

$$K_\alpha^{g_L} = \{x : \lambda(x) = \alpha\}.$$

We obtain two multifractal spectra $\mathcal{L}_D$ and $\mathcal{L}_E$ specified, respectively, by the pairs of functions (g_L, G_D) and (g_L, G_E), where the set functions G_D and G_E are given by (4) and (5). We call them *multifractal spectra for Lyapunov exponents.* The spectrum $\mathcal{L}_D$ was studied in [54] (see also the references in that paper). The spectrum $\mathcal{L}_E$ was introduced in [14].

We want to state the fundamental theorem for some spectra for expanding conformal repellers. The way we state it it is a conglomeration of results which accumulated the last years by an enormous amount of literature. For a positive Hölder continuous function $u: \Sigma \to \mathbb{R}^+$ we define in a unique way a metric ρ_u on Σ. This metric is specified by the condition

$$\operatorname{diam}_u(C_n(x)) = \exp\left\{-\sum_{k=0}^{n-1} u(\sigma^k x)\right\}.$$

We denote by $\dim_u(Z)$, $d_{\mu,u}(x)$ the dimensions w.r.t. the metric ρ_u. If we choose $u \equiv 1$ then the u–dimensions are simply the entropies and for $u = \log a$ we recover the dimensions on the expanding conformal repeller (see [3]).

For every real number α, set

$$K_\alpha = \{x \in \Sigma \colon d_{\mu,u}(x) = \alpha\}.$$

We set

$$\mathcal{D}_u(\alpha) = \dim_u K_\alpha.$$

The function $\alpha \mapsto \mathcal{D}_u(\alpha)$ is called the **u-dimension spectrum for u-pointwise dimensions** (with respect to the measure μ). Let φ be a continuous function on Σ. For every real number q, we define the function

$$\varphi_q = -T_u(q)u + q\varphi,$$

where the number $T_u(q)$ is chosen such that $P(\varphi_q) = 0$. We denote by ν_q and m_u, respectively, the equilibrium measures of φ_q and $-\dim_u \Sigma \cdot u$ with respect to σ.

The following is a complete multifractal analysis of the spectrum $\mathcal{D}_u$ for subshifts of finite type.

Theorem 13. *Let $\sigma|\Sigma$ be a one-sided or two-sided topologically mixing subshift of finite type, u and φ Hölder continuous functions on Σ, such that u is positive and $P(\varphi) = 0$, and μ the equilibrium measure of φ with respect to σ. Then, the following properties hold:*

1. *For μ-almost every $x \in \Sigma$, the u-pointwise dimension of μ at x exists and*
$$d_{\mu,u}(x) = -\frac{\int_\Sigma \varphi \, d\mu}{\int_\Sigma u \, d\mu} = \frac{h_\mu(\sigma)}{\int_\Sigma u \, d\mu}.$$
2. *The function $q \mapsto T_u(q)$ is real analytic on $\mathbb{R}$, and satisfies $T_u'(q) \le 0$ and $T_u''(q) \ge 0$ for every $q \in \mathbb{R}$. Moreover, $T_u(0) = \dim_u \Sigma$ and $T_u(1) = 0$.*
3. *The domain of the function $\alpha \mapsto \mathcal{D}_u(\alpha)$ is a closed interval in $[0, +\infty)$ and coincides with the range of the function $\alpha_u(q) = -T_u'(q)$. For every $q \in \mathbb{R}$, we have*
$$\mathcal{D}_u(\alpha_u(q)) = T_u(q) + q\alpha_u(q),$$
and
$$\alpha_u(q) = -\frac{\int_\Sigma \varphi \, d\nu_q}{\int_\Sigma u \, d\nu_q}.$$
4. *For every $q \in \mathbb{R}$, $\nu_q(K_{\alpha_u(q)}) = 1$, and*
$$d_{\nu_q,u}(x) = T_u(q) + q\alpha_u(q)$$
for ν_q-almost all $x \in K_{\alpha_u(q)}$. Moreover, $\overline{d}_{\nu_q,u}(x) \le T_u(q) + q\alpha_u(q)$ for every $x \in K_{\alpha_u(q)}$, and $\mathcal{D}_u(\alpha_u(q)) = \dim_u \nu_q$ for every $q \in \mathbb{R}$.

5. *If $\mu \neq m_u$, then $\mathcal{D}_u$ and T_u are real analytic strictly convex functions, and $(\mathcal{D}_u, T_u)$ is a Legendre pair with respect to the variables α, q.*
6. *If $\mu = m_u$, then $d_{\mu,u}(x) = \dim_u \Sigma$ for every $x \in \Sigma$.*

We showed in [6,7] in the case of expanding conformal repellers or hyperbolic horseshoes on surfaces that if, for instance, the spectra $\mathcal{D}_D$ and $\mathcal{E}_E$ are equivalent then all other spectra are equivalent. We call this phenomenon *multifractal rigidity.* It indicates that the spectra $\mathcal{D}_D$ and $\mathcal{E}_E$ are essentially independent. Moreover, both spectra for the Lyapunovexponent can be recovered from the entropy or dimension spectrum of an appropriately chosen measure.

7 Multifractal rigidity

In [6] we considered another interesting phenomenon in dimension theory of dynamical systems which we regard as a multifractal rigidity phenomenon. Roughly speaking, it states that if two dynamical systems are topologically equivalent (via a homeomorphism) and some of their multifractal spectra coincide, then they are smoothly equivalent (via a diffeomorphism). This leads to a classification of maps and Gibbs measures using multifractal spectra and/or multifractal decompositions. We believe that this type of classification fits well with the "physical" interpretation of the equivalence of dynamical systems. We think that this is a non-trivial and challenging problem.

If $\mathcal{F}$ and $\widehat{\mathcal{F}}$ are two multifractal spectra for the maps f and $\widehat{f}$, respectively, the condition $\mathcal{F} = \widehat{\mathcal{F}}$ indicates the existence of a "symmetry" between the two dynamical systems. Thus, it is a requirement of "physical" nature, and this multifractal rigidity has a strong physical content: if the spectra of two dynamical systems are equal, then the systems are the same up to a change of variables, and thus should be considered the same from the physical point of view.

7.1 Preservation of the spectra under Markov coding

In the theory of hyperbolic dynamical systems one often uses Markov partitions to encode the system. The coding space is a symbolic dynamical system and several useful tools from symbolic dynamics can be exploited. Among a long list of results in this direction thermodynamic formalism is one of the most fruitful tools. It is used to proof the existence of Gibbs states (see f.e. [11]) and to derive their multifractal properties (see f.e. [30,32,54,6,7]).

One of the problems of applying the coding method is that the coding map is not one–to–one on the boundaries of the Markov partitions. In particular, it is not obvious that sets of positive measure are mapped to sets of positive measure and that dimensionlike notions are preserved. The same problem occurs if one deals with entropy. In the above mentioned applications it was sufficient to establish the fact that the boundary of the Markov partition (see [11]) has zero measure for any Gibbs state. But there are problems when one has to deal with multifractal decompositions. There we have to cope with the fact that the boundaries of the Markov partitions might have positive entropy or dimensions. Therefore the coding could a priori change the spectrum. In [44] we proved:

Theorem 14. *If Λ is a horseshoe on a surface (J an expanding conformal repeller) then the map $\chi\colon \Sigma_A \to \Lambda$ ($\chi\colon \Sigma_A^+ \to J$) preserves for any subset $Z \in \Sigma_A$ ($Z \in \Sigma_A^+$) its Hausdorff dimension and its topological entropy.*

Remark 15. In [44] it is shown that the statement is no longer true for piecewise smooth systems (even with nice Markov partitions). The coding map can strictly decrease entropy and dimension.

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