Multi-pulse homoclinic loops in systems with a smooth first integral

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Abstract. We prove that the orbit-flip bifurcation in the systems with a smooth first integral (e.g. in the Hamiltonian ones) leads to appearance of infinitely many multi-pulse self-localized solutions. We give a complete description to this set in the language of symbolic dynamics and reveal the role played by special non-selflocalized solutions (e.g. periodic and heteroclinic ones) in the structure of the set of self-localized solutions. We pay a special attention to the superhomoclinic ("homoclinic to homoclinic") orbits whose presence leads to a particularly rich structure of this set.

1 Introduction

Consider a $2n$-dimensional ($n \geq 2$) dynamical system

$$\dot{x} = X(x)$$

with a smooth first integral H, i.e.,

$$H'(x)X(x) \equiv 0. \tag{1}$$

A Hamiltonian system with n degrees of freedom is a natural example but the symplectic structure is not important for our purposes.

Let X have a hyperbolic equilibrium state O at the origin (i.e. $X(0) = 0$ and the eigenvalues of the matrix $X'(0)$ do not lie on the imaginary axis). By (1)

$$H'(0)X'(0) = 0$$

so, since $X'(0)$ is non-degenerate by assumption, the linear part of H at O vanishes. Assume that the quadratic part of H at O is *a non-degenerate quadratic form*. It is an easy exercise to check that when this non-degenericity assumption holds, the system near O may be brought by a linear transformation of coordinates to the following form

$$\dot{u} = -Bu + \ldots, \quad \dot{v} = B^{\top}v + \ldots \tag{2}$$

where $u \in R^n$, $v \in R^n$, the dots stand for nonlinearities and B is a matrix whose eigenvalues have positive real parts. Moreover, the first integral takes the form

$$H = (v, Bu) + \ldots \tag{3}$$

where the dots stand for the third and higher order terms.

Let $\lambda_1, \dots \lambda_n$ be the eigenvalues of B, ordered in such way that $0 < \mathrm{Re}\lambda_1 \le \dots \le \mathrm{Re}\lambda_n$. We assume that the first two leading eigenvalues of B are real and different; precisely, we assume

$$0 < \lambda_1 < \lambda_2 < \mathrm{Re}\lambda_i \qquad (i > 2).$$

In this case the matrix B may be written in the form

$$B = \begin{pmatrix} \lambda_1 & 0 & \\ & & O \\ 0 & \lambda_2 & \\ & & \\ O & & B^0 \end{pmatrix} \tag{4}$$

where the real parts of the eigenvalues of B^0 are strictly greater than λ_2.

The equilibrium state O is a saddle with n-dimensional stable and unstable manifolds W^s_O and W^u_O which are tangent at O to the u-space and v-space, respectively. Both the invariant manifolds lie in the $(2n-1)$-dimensional level $\{H = 0\}$ and they may intersect *transversely* in that level, producing a number of homoclinic loops, i.e. the orbits which tend to O both as $t \to +\infty$ and $t \to -\infty$ (see Fig.1). This paper addresses the question on the possible structure of the homoclinic loops in the given class of systems, in particular, on the conditions for the coexistence of infinitely many of homoclinic loops.

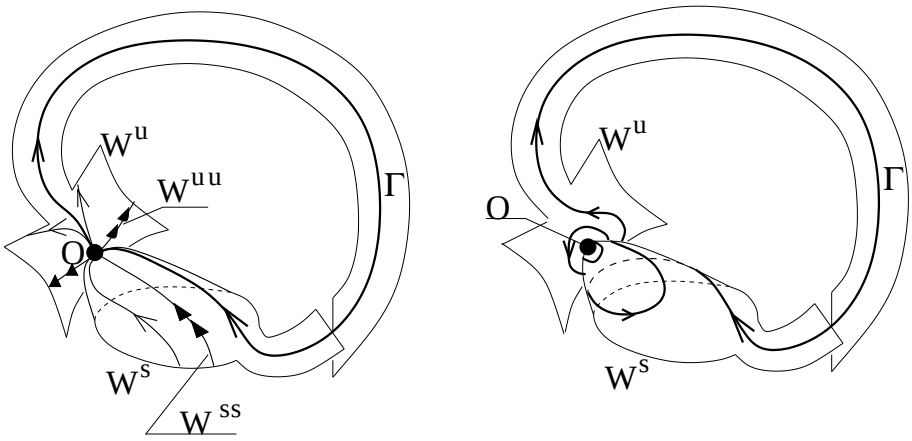

Fig. 1. A homoclinic orbit Γ of a transverse intersection of the stable and unstable manifolds of a saddle (left) or a saddle-focus (right) O.

It follows from [1–3] that (generically) there exists infinitely many homoclinic loops in an arbitrarily small neighborhood of a single homoclinic loop

to a saddle-focus (this is the case where λ_1 and λ_2 are a pair of complex-conjugate numbers, we do not consider this case in this paper). On the contrary, when the equilibrium state is a saddle (i.e. λ_1 is real) no other homoclinic loops can accumulate to a homoclinic loop in general position [4]. The homoclinic loops correspond to self-localized (decaying to zero as $t \to \pm\infty$) solutions of (1). When O is a saddle, this solution tends to zero monotonically in time whereas the time dependence of any component of the self-localized solution is, typically, oscillatory when O is a saddle-focus. Thus, the cited results suggest that a self-localized solution with oscillatory tales is accompanied by infinitely many multi-pulse solutions, and self-localized solutions with monotonic tales do not form infinite series, generically. This contradicts to the fact that plenty of multi-pulse solutions with monotonic tales have been seen in different Hamiltonian systems.

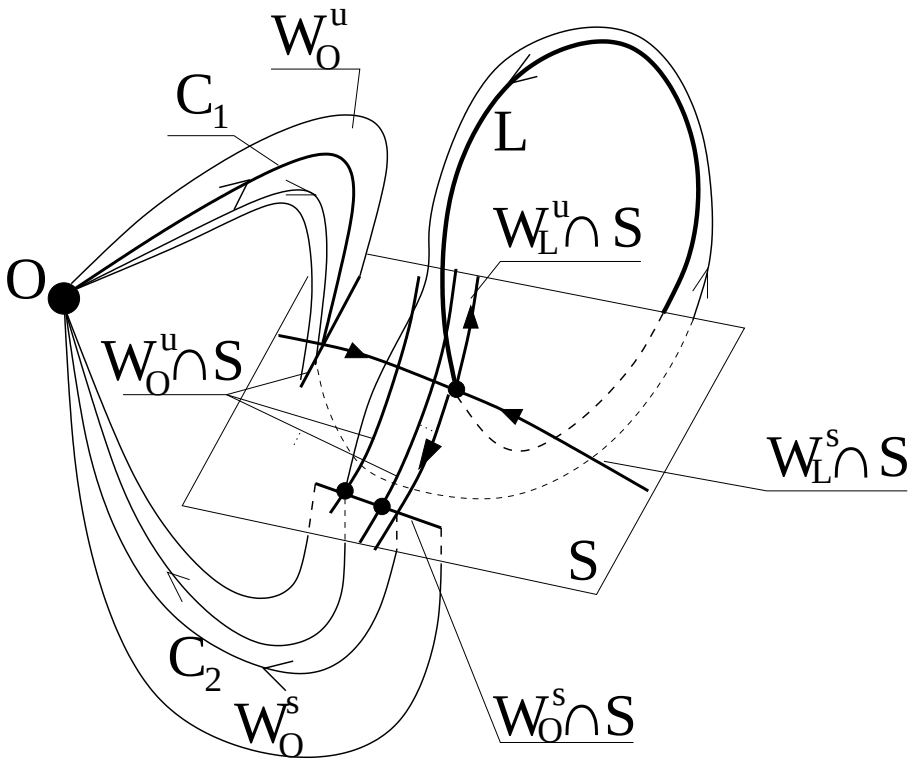

Fig. 2. Infinitely many homoclinic loops appear as a result of a transverse intersection of the invariant manifolds of O and a saddle periodic orbit L.

To resolve this problem, a simple scenario of appearance of infinitely many homoclinic loops to a saddle was proposed in [4]: if a saddle periodic orbit L

exists in the zero level of the first integral ($L \in \{H = 0\}$) and if the unstable manifold of the saddle O intersects transversely the stable manifold of L whereas the unstable manifold of L intersects transversely the stable manifold of O, then infinite sequence of homoclinic loops exists which accumulate to the union of O, L and the pair of heteroclinic connections. This statement is a simple consequence of λ-lemma: take a small cross-section S to L in $\{H = 0\}$; since $W^u_O \cap S$ intersects $W^s_L \cap S$ transversely, the infinite sequence of images of $W^u_O \cap S$ by the Poincaré map near L accumulates to $W^s_L \cap S$; each of these images must, hence, intersect $W^s_O \cap S$ transversely (as $W^s_L \cap S$ does so by assumption), producing thereby a homoclinic orbit (Fig.2).

We start this paper with showing how such configuration appears at the so-called orbit-flip bifurcation of the homoclinic loop[1]. Namely, let the system have a transverse homoclinic loop Γ. We assume that Γ enters O as $t \to +\infty$ along *the leading direction*, i.e. it is tangent at O at $t = +\infty$ to that eigenvector of B in the u-space which corresponds to the eigenvalue λ_1. On the contrary, we require that at $t = -\infty$,
the homoclinic orbit Γ leaves O along the eigenvector of $B^\top$ in the v-space which corresponds to the eigenvalue λ_2 (the next after leading).

Note that the situation we consider here is essentially irreversible, so our orbit-flip bifurcation is different in many instances from those considered earlier in the reversible case [5,6].

The trajectories in the unstable manifold which leave O not along the leading direction form a smooth $(n-1)$-dimensional submanifold W^{uu} of W^u, transverse to the leading direction and tangent at O to the invariant subspace (in the v-space) of the matrix $B^\top$ which corresponds to the eigenvalues $\lambda_2, \dots, \lambda_n$. The above assumption implies that $\Gamma \subset W^{uu}$. The presence of a common orbit of the n-dimensional manifold W^s and the $(n-1)$-dimensional manifold W^{uu} both lying in the $(2n-1)$-dimensional hypersurface $\{H = 0\}$ is an event of codimension one. By a small perturbation of the system (not moving it out of the class of systems with a smooth first integral) the orbit of homoclinic intersection of W^u and W^s will, generically, miss W^{uu}. To study this bifurcation we will embed our system (1) in a one-parameter family of systems with a smooth first integral, depending continuously on a parameter μ (the first integral H is assumed to depend continuously on μ as well). The original system will correspond to $\mu = 0$ and we consider the bifurcations at small μ. The system will retain its form (2), (4) (with the formula (3) still valid for H) where $\lambda_{1,2}$ and B^0 are now continuous functions of μ (as well as the terms denoted by dots in (2),(3) are).

[1] Note that the orbit-flip is the only codimension-1 homoclinic bifurcation in the class of systems with a first integral which could give rise to the birth of infinite series of multi-pulse self-localized solutions with monotonic tales (the two other codimension-1 bifurcations - the tangency of stable and unstable manifolds and the transition from a saddle to a saddle-focus - are known to produce no non-oscillating multi-pulse loops).

Since the manifolds W^s and W^u depend on μ continuously and their intersection along Γ is transverse at $\mu = 0$, this intersection persists at small μ and the corresponding homoclinic orbit Γ_μ depends on μ continuously. We assume that $\Gamma_\mu \not\subseteq W^{uu}$ at $\mu \neq 0$; moreover $\Gamma_\mu \subseteq W^{u+}$ at $\mu > 0$ and $\Gamma_\mu \subseteq W^{u-}$ at $\mu < 0$ where W^{u+} and W^{u-} denote the two connected components into which W^{uu} divides W^u (Fig.3).

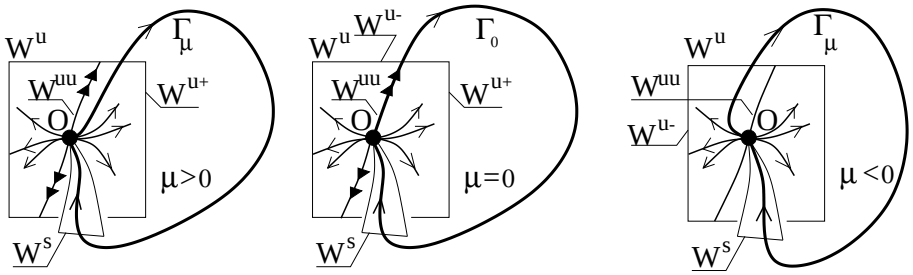

Fig. 3. The orbit-flip bifurcation: at $\mu = 0$ the homoclinic orbit Γ lies in the strong-unstable manifold of the saddle O.

Theorem 1 in the next Section shows that, generically, a saddle periodic orbit $L \in \{H = 0\}$ is born from Γ as μ passes through zero and this indeed implies the birth of infinitely many multi-pulse homoclinic loops. In the same Section we also analyze how the general structure of the set of homoclinic loops is changed due to the orbit-flip bifurcation. Namely, we establish that if a homoclinic loop $\tilde{\Gamma}$ in general position exists simultaneously with the bifurcating loop Γ, then either a double homoclinic loop close to a concatenation $\tilde{\Gamma}\Gamma$ or an infinite family of loops close to $\tilde{\Gamma}\Gamma^k$ $(k = 1, \ldots, \infty)$ is born as μ passes through zero (see theorems 2,3).

Far richer possibilities are opening when we include in the picture the so-called *superhomoclinic* (i.e. "homoclinic to homoclinic") orbits. Like the existence of a homoclinic orbit to a single periodic orbit implies the existence of infinitely many periodic orbits [7], the existence of an orbit which is homoclinic to a single homoclinic loop may imply the existence of infinitely many of loops. We show in Section 3 that at the moment of the orbit-flip bifurcation in the so-called orientable case the homoclinic loop Γ has *the unstable manifold* $W^u_\Gamma \in \{H = 0\}$ which is a smooth n-dimensional manifold with a boundary (the boundary is the manifold W^{uu}) which consists of the orbits whose limit set as $t \to -\infty$ is Γ. This manifold is the limit of the unstable manifold of the periodic orbit L_μ which tends to Γ as $\mu \to 0$ (the stable manifold of L_μ tends to the stable manifold of O). Since W^s_Γ is n-dimensional and since it lies, as a whole, in the $(2n-1)$-dimensional level $\{H = 0\}$, it may intersect transversely with W^s_O. Here, we call the orbits of such intersection the superhomoclinic orbits (see Fig.4). We show that their presence implies

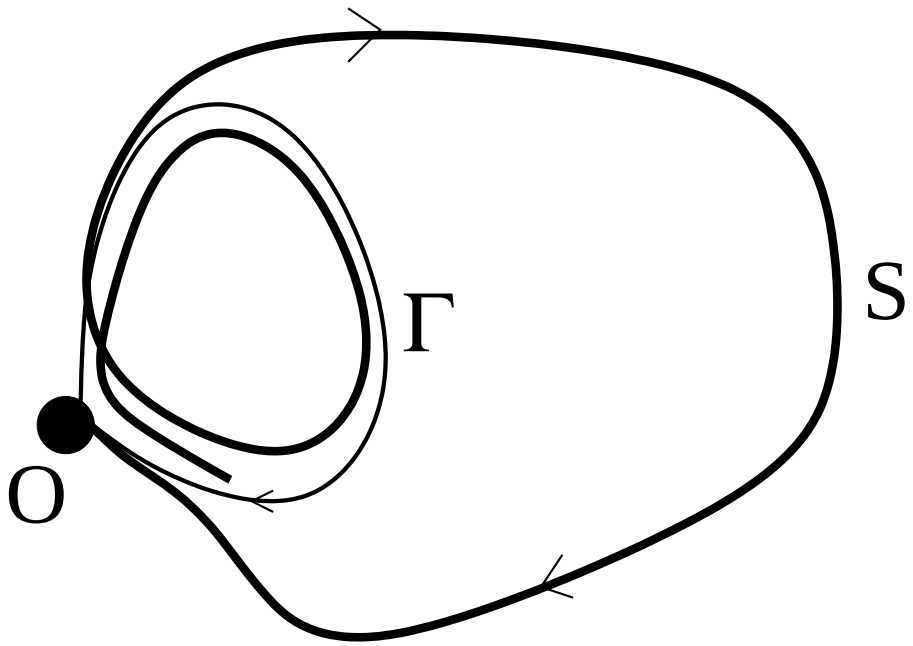

Fig. 4. A supehomoclinic orbit S is α-limit to the homoclinic loop Γ and ω-limit to the saddle O.

immediately the existence of an infinite set of multi-pulse homoclinic loops with a nontrivial structure.

Bifurcations of superhomoclinic orbits in general (non-Hamiltonian) systems were studied in [8,9] (some cases were considered earlier in [10–12]). For systems with the smooth first integral, superhomoclinic orbits were discovered in [13] (the proofs are in [14]) in connection with the problem of the explanation of the existence of infinitely many self-localized solutions in an applied problem. Our construction here is close to that studied in [8,9] and it is quite different from that in [13,14]. However, the main idea remains the same: superhomoclinic orbits seem to play a major role in organizing the set of multi-pulse homoclinic loops in Hamiltonian systems

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2 Orbit-flip bifurcation

We impose, first, some genericity assumptions on the system under consideration, which are necessary to study the orbit-flip bifurcation. The first two of them were the transversality of the intersection of W^s and W^u along Γ and the requirement that $\Gamma \notin W^{ss}$ (i.e. it enters O as $t \to +\infty$ along the leading direction).

To formulate the third genericity assumption we recall (see [15]) that an extended stable manifold W^{see} which is a smooth $(n+2)$-dimensional invariant manifold tangent at O to the direct sum of the u-space and the invariant subspace of $B^\top$ in the v-space which corresponds to the leading eigenvalues λ_1 and λ_2. Note that W^{see} contains the stable manifold W^s, so it contains the homoclinic orbit Γ (note that W^{see} is not unique but any two of such manifolds are tangent to each other at every point of W^s). We require that at $\mu = 0$, at the points of Γ the manifold W^{see} is transverse to the strong unstable manifold W^{uu} (by invariance of W^{see} and W^{uu} it is sufficient to require the transversality at an arbitrary single point on Γ).

According to [16] this kind of transversality assumption is sufficient for the result of [17] to be fulfilled; namely, it guarantees the existence of a C^1-smooth invariant repelling $(n+2)$-dimensional manifold which is transverse to W^{uu} and which contains all orbits staying in a small neighborhood of the homoclinic loop Γ for all times.

The fourth genericity assumption is

$$\lambda_2 \neq 2\lambda_1.$$

It is not a technical assumption; we will see that the cases $\lambda_2 < 2\lambda_1$ and $\lambda_2 > 2\lambda_1$ are indeed different (though the results are similar). We will also need a different smoothness assumptions in these cases: the system will be assumed C^r-smooth with $r \geq 3$ at $\lambda_2 < 2\lambda_1$ and $r \geq 4$ at$\lambda_2 > 2\lambda_1$.

Most importantly, the last, fifth, genericity assumption is different in the cases $\lambda_2 < 2\lambda_1$ and $\lambda_2 > 2\lambda_1$. If $\lambda_2 < 2\lambda_1$, then in W^s there exists a special smooth (at least C^2) invariant $(n-1)$-dimensional manifold W^{s0} which is tangent at O to the eigenspace of the matrix B in the u-space which corresponds to the eigenvalues $\lambda_1, \lambda_3, \dots, \lambda_n$ (i.e. it is transverse to the eigenvector corresponding to the eigenvalue λ_2). The existence of this manifold is proved later. We will assume that in this case

$$\Gamma \not\subset W^{s0}.$$

Basically, this means that when Γ enters O at $t = +\infty$, the coordinate u_1 (the projection onto the stable leading eigenvector) behaves asymptotically as

$$u_{11}e^{-\lambda_1 t} + u_{12}e^{-\lambda_2 t} + O(e^{-2\lambda_1 t})$$

where $u_{12} \neq 0$ (the non-vanishing of u_{11} is given by the assumption $\Gamma \not\subset W^{ss}$).

When $\lambda_2 > 2\lambda_1$, the special manifold W^{u0} is not defined uniquely and, moreover, the above assumption is unnecessary. An important requirement we need in this case is that

the intersection of the extended unstable manifold W^{ue} with the stable manifold W^s along Γ is transverse in R^{2n}.

This extended unstable manifold is an $(n+1)$-dimensional smooth invariant manifold which is tangent at O to the direct sum of the v-space and the leading eigenvector in the u-space (it is the eigenvector corresponding to the leading eigenvalue λ_1 of B) (see [15]). This manifold is not unique but any two of them contain the stable manifold W^u and are tangent to each other at every point of W^u. Hence, the transversality assumption above is well posed (recall that $\Gamma \subset W^u$). Note that we speak here about the transversality in the whole phase space, not in the level $\{H = 0\}$. The intersection of W^{ue} with $\{H = 0\}$ is the union of two n-dimensional manifolds: one is W^u and the second is a smooth manifold W^{u1} which intersects W^u at the points of the manifold W^{uu} transversely in $\{H = 0\}$. Since $\Gamma_\mu \subset W^{uu}$ at $\mu = 0$, the above transversality assumption can be read as the transversality (in $\{H = 0\}$) of the intersection of W^{u1} and W^s along the homoclinic loop Γ_0. Note that this requirement is unnecessary if $\lambda_2 < 2\lambda_1$.

In both cases, the fifth non-degeneracy assumption can be expressed as a non-vanishing of some functional $A(X)$ which will be explicitly defined later. We will introduce also a functional $a(X)$ whose non-vanishing is equivalent to the transversality of W^u and W^s. The signs of A and a determine the structure of bifurcations which happened at $\mu \neq 0$.

Theorem 1. *Let U be a sufficiently small neighborhood of Γ_0 in the level $\{H = 0\}$. At $Aa\mu \geq 0$ there is no other orbit, except for Γ_μ and O, which stays in U for all times. At $Aa\mu < 0$, the set of the orbits staying in U for all times consists of: O, Γ_μ, a single-round periodic orbit L_μ, a pair of heteroclinic orbits $C_{1\mu}$ and $C_{2\mu}$ - the former is α-limit to O and ω-limit to*

L_μ whereas the latter is α-limit to L_μ and ω-limit to O, and a sequence of homoclinic loops $\Gamma_{k\mu}$ ($\Gamma_{k\mu}$ is a k-round loop, $k = 2, \ldots$, one such loop for each k) which accumulate to the union $O \cup L_\mu \cup C_{1\mu} \cup C_{2\mu}$.

Generically, in addition to Γ, the system at $\mu = 0$ may have some number of other homoclinic loops $\Gamma_+^1, \ldots, \Gamma_+^{m_+}$ and $\Gamma_-^1, \ldots, \Gamma_-^{m_-}$ which correspond to transverse intersection of W^s and W^u and which do not lie neither in W^{uu} nor in W^{ss} (i.e. they leave and enter O along the leading directions). We assume that the loops $\Gamma_+^1, \ldots, \Gamma_+^{m_+}$ lie in W^{s+} and the loops $\Gamma_-^1, \ldots, \Gamma_-^{m_-}$ lie in W^{s-} where $W^{s\pm}$ are two components into which W^{ss} divides W^s: we assume that the orbit Γ_μ belongs to W^{s+}.

Let U be a small neighborhood of the homoclinic bunch $\Gamma \cup \Gamma_+^1 \cup \ldots \cup \Gamma_+^{m_+} \cup \Gamma_-^1 \cup \ldots \cup \Gamma_-^{m_-} \cup O$ in the level $\{H = 0\}$. It is a union of a small neighborhood of O with $m_+ + m_- + 1$ handles U_0, U_{1+}, ..., U_{m_++}, U_{1-}, ..., U_{m_--} (the handle U_0 surrounds Γ). Since the fundamental group of U is nontrivial, every orbit in U gets its natural coding which describes the sequence of handles the orbit visits as time runs. Thus, the coding of O is the empty sequence, Γ is coded by 0, the loops $\Gamma_\pm^i$ are coded by $i\pm$ respectively, the periodic orbit L_μ from theorem 1 is coded by the infinite sequence of 0's, the heteroclinic orbits $C_{1\mu}$ and $C_{2\mu}$ are coded, respectively, by the infinite to the right and infinite to the left sequences of 0's; the k-round homoclinic loops from theorem 1 are coded by 0^k.

Theorem 2. *Except for the orbits given by theorem 1 and the homoclinic loops $\Gamma_\pm^i$, the set of all orbits lying entirely in U contains the following orbits (and only them): double homoclinic loops $(i-)0$ (where $i = 1, \ldots, m_-$) at $Aa\mu > 0$; nothing at $\mu = 0$; exactly one homoclinic loop $(i+)0^k$ for each $k \geq 1$ and $i = 1, \ldots, m_+$, and m_+ heteroclinic connections $(1+)0^\infty$, ..., $(m_++)0^\infty$ from O to L_μ at $Aa\mu < 0$ (as $k \to +\infty$, the limit of the sequence of loops $(i+)0^k$ is the heteroclinic connection $(i+)0^\infty$).*

Let us prove theorems 1 and 2. Choose the coordinates $(u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n)$ near O such that the u_1-axis will be the eigenvector of B corresponding to the leading eigenvalue λ_1, the u_2-axis will be the eigenvector of B corresponding to the next eigenvalue λ_2 and the plane $(u_1 = u_2 = 0)$ will be the eigenspace corresponding to the rest of the spectrum of B; similarly, let the v_1-axis be the eigenvector of $B^\top$ corresponding to λ_1, the v_2-axis be the eigenvector of $B^\top$ corresponding to λ_2 and the plane $(v_1 = v_2 = 0)$ be the eigenspace corresponding to the rest of the spectrum of $B^\top$. By assumption, Γ enters O at $t = +\infty$ tangent to the u_1-axis. We choose the sign of u_1 such that $u_1 > 0$ on Γ at t close to $+\infty$; i.e. the component W^{s+} of W^s corresponds to the positive direction of the u_1-axis. At $\mu = 0$ the homoclinic orbit Γ is tangent at O to the v_2-axis at $t = -\infty$. We assume that $v_2 > 0$ on Γ at t close to $-\infty$. Moreover, we assume that Γ adjoins O at $t = -\infty$ from the side of positive v_1 at $\mu > 0$ and from the side of negative v_1 at $\mu < 0$; i.e. the component

W^{u+} extends from W^{uu} towards $v_1 > 0$ and W^{u-} extends towards negative v_1.

Let us straighten the invariant manifolds W^s and W^u near O so that their equations will be, respectively, $v = 0$ and $u = 0$ locally. The system will take the following form near O:

$$\dot u = -Bu + f(u,v)u, \quad \dot v = B^\top v + g(u,v)v \tag{5}$$

where f and g are some C^{r-1}-functions vanishing at zero. The first integral is now locally written as

$$H = (v, Bu) + H_0(u,v) \tag{6}$$

where H_0 vanish identically both at $u = 0$ and $v = 0$. According to [18] (see also [19,12] and [15]), by an additional C^{r-1}-smooth transformation of coordinates system (5) is brought to the following form, where we denote $u^0 = (u_3, \dots, u_n)$ and $v^0 = (v_3, \dots, v_n)$:

$$\begin{aligned}
\dot u_1 &= -\lambda_1 u_1 + f_{11}(u_1, v)u_1 + f_{12}(u_1, u_2, v)u_2 + f_{10}(u,v)u^0,\\
\dot u_2 &= -\lambda_2 u_2 + f_{21}(u_1, v)u_1 + f_{22}(u_1, u_2, v)u_2 + f_{20}(u,v)u^0,\\
\dot u^0 &= -B^0 u^0 + f_{01}(u_1, v)u_1 + f_{02}(u_1, u_2, v)u_2 + f_{00}(u,v)u^0,\\
\dot v_1 &= \lambda_1 v_1 + g_{11}(u, v_1)v_1 + g_{12}(u, v_1, v_2)v_2 + g_{10}(u,v)v^0,\\
\dot v_2 &= \lambda_2 v_2 + g_{21}(u, v_1)v_1 + g_{22}(u, v_1, v_2)v_2 + g_{20}(u,v)v^0,\\
\dot v^0 &= (B^0)^\top v^0 + g_{01}(u, v_1)v_1 + g_{02}(u, v_1, v_2)v_2 + g_{00}(u,v)v^0
\end{aligned} \tag{7}$$

with the C^{r-1}-functions f_{ij}, g_{ij} vanishing at zero and satisfying the following identities

$$\begin{array}{llll}
f_{i1}(0,v) \equiv 0, & g_{i1}(u,0) \equiv 0 & (i = 1,2,0), & \\
f_{i2}(0,0,v) \equiv 0, & g_{i2}(u,0,0) \equiv 0 & (i = 2,0), & \\
f_{11}(u_1,0) \equiv 0, & f_{12}(u_1,u_2,0) \equiv 0, & f_{10}(u,0) \equiv 0, & \\
g_{11}(0,v_1) \equiv 0, & g_{12}(0,v_1,v_2) \equiv 0, & g_{10}(0,v) \equiv 0 &
\end{array} \tag{8}$$

and, at $\lambda_2 < 2\lambda_1$, the following additional identities

$$\begin{array}{llll}
f_{12}(0,0,v) \equiv 0, & g_{12}(u,0,0) \equiv 0, & & \\
f_{21}(u_1,0) \equiv 0, & f_{22}(u_1,u_2,0) \equiv 0, & f_{20}(u,0) \equiv 0, & \\
g_{21}(0,v_1) \equiv 0, & g_{22}(0,v_1,v_2) \equiv 0, & g_{20}(0,v) \equiv 0. &
\end{array} \tag{9}$$

By [12], an additional C^{r-2}-smooth coordinate transformation can be done in the case $\lambda_2 > 2\lambda_1$ which keeps the system in the form (7),(8) with f_{ij}, g_{ij}

(now C^{r-2}) satisfying the following additional identities:

$$\begin{aligned} \frac{\partial f_{1j}}{\partial v_1} &\equiv 0 \quad \text{at} \quad v_1 = 0 \\ \frac{\partial g_{1j}}{\partial u_1} &\equiv 0 \quad \text{at} \quad u_1 = 0. \end{aligned} \tag{10}$$

Hereafter we assume that the system is brought to this form. We denote the smoothness of the obtained system as q (i.e. $q = r - 1$ at $\lambda_2 < 2\lambda_1$ and $q = r - 2$ at $\lambda_2 > 2\lambda_1$, so $q \geq 2$ in both cases).

In these coordinates, the non-leading manifolds W^{ss} and W^{uu} are given by equations $\{v = 0, u_1 = 0\}$ and $\{u = 0, v_1 = 0\}$, respectively. Furthermore, identities (8) guarantee that the extended unstable manifold W^{ue} is tangent to $\{u_2 = 0, u^0 = 0\}$ at the points of the local unstable manifold $W^u_{\rm loc} : \{u = 0\}$. Indeed, the tangents to W^{ue} at the points $W^u_{\rm loc}$ form a continuous field of linear spaces invariant with respect to the flow linearized along the orbits in $W^u_{\rm loc}$ and this field is transverse to W^{ss} at O. According to [15] such field is unique. When identities (8) are satisfied, the space $\{u_2 = 0, u^0 = 0\}$ is invariant with respect to the linearized flow and it is transverse to W^{ss} at O, hence it is the tangent to W^{ue} indeed. Thus, W^{ue} is locally given by an equation of the form

$$(u_2, u^0) = h^{ue}(u_1, v) \tag{11}$$

where h^{ue} vanishes at zero along with its first derivatives. Note that h^{ue} must vanish identically at $u_1 = 0$ because W^{ue} contains $W^u_{\rm loc} : \{u = 0\}$ by definition. Now, it is seen that the first integral (6) on $W^{ue}_{\rm loc}$ is written in the form

$$H = \lambda_1 u_1 (v_1 - h^{u1}(u_1, v))$$

for some smooth h^{u1} which vanish at zero along with the first derivative. Hence, the intersection $W^{ue}_{\rm loc} \cap \{H = 0\}$ is the union of $W^u_{\rm loc}$ and a C^1-manifold $W^{u1}_{\rm loc}$ given by (11) with the constraint

$$v_1 = h^{u1}(u_1, v). \tag{12}$$

The intersection of W^{u1} with $W^u_{\rm loc}$ must be an $(n-1)$-dimensional invariant submanifold of $W^u_{\rm loc}$, transverse to the v_1-axis in virtue of (12). Such a submanifold is unique – it is $W^{uu}_{\rm loc}$. Thus,

$$W^{u1}_{\rm loc} \cap W^u_{\rm loc} = W^{uu}_{\rm loc},$$

i.e. $h^{u1}(0, v) \equiv 0$.

Analogously, the tangent to W^{see} at the points of $W^s_{\rm loc}$ is $v^0 = 0$.

When $\lambda_2 < 2\lambda_1$, identities (8), (9) imply that the evolution of the variables (u_1, u_2) on $W^s_{\rm loc}$ is independent on u^0 and is governed by the linear system

$$\dot{u}_1 = \lambda_1 u_1, \qquad \dot{u}_2 = \lambda_2 u_2.$$

Thus, for every orbit in $W^s_{\rm loc}\backslash W^{ss}$ we have $u_2(t)=Cu_1^\alpha(t)$ with $\alpha=\lambda_2/\lambda_1<2$. It follows that $u_2=0$ is a unique invariant submanifold of $W^s_{\rm loc}$ which is transverse to the u_2-axis and which is at least C^2-smooth. We denote this manifold as W^{s0}.

Take a small $d>0$ and consider a pair of $(2n-2)$-dimensional cross-sections $\Pi^{\rm in}$ and $\Pi^{\rm out}$ to the homoclinic loop Γ: $\Pi^{\rm in}=\{u_1=d\}\cap\{H=0\}$ and $\Pi^{\rm out}=\{v_2=d\}\cap\{H=0\}$. Let $M^{\rm in}(u^{\rm in},v^{\rm in})=\Gamma\cap\Pi^{\rm in}$ and $M^{\rm out}(u^{\rm out},v^{\rm out})=\Gamma\cap\Pi^{\rm out}$. Since $M^{\rm in}\in W^s_{\rm loc}$ and $M^{\rm out}\in W^u_{\rm loc}$, it follows that $v^{\rm in}\equiv 0$ and $u^{\rm out}\equiv 0$. By assumption, $M^{\rm out}\in W^{uu}_{\rm loc}$ at $\mu=0$, therefore $v_1^{\rm out}|_{\mu=0}=0$. When μ increases through zero, the value of $v_1^{\rm out}$ changes from negative values to positive, so we may simply assume

$$v_1^{\rm out}=\mu. \tag{13}$$

Recall that $v_2^{\rm out}=u_1^{\rm in}=d$. Since $M^{\rm in}\notin W^{s0}$ at $\lambda_2<2\lambda_1$, it follows that

$$u_2^{\rm in}\neq 0 \text{ in the case } \lambda_2<2\lambda_1. \tag{14}$$

We take a small $\delta>0$ and shrink $\Pi^{\rm in}$ and $\Pi^{\rm out}$ to the size δ neighborhoods of $M^{\rm in}$ and $M^{\rm out}$, respectively. In particular, we have $\|v^0-v^{0\rm out}\|\le\delta$ on $\Pi^{\rm out}$. Since the orbit Γ is tangent to the v_2-axis at $\mu=0$ by assumption, it follows that

$$\|v^{\rm out}\|\ll d$$

on $\Pi^{\rm out}$.

Orbits which lie in the level $\{H=0\}$ in a small neighborhood of Γ must intersect $\Pi^{in,out}$, so the problem of the study of these orbits reduces to the study of the Poincaré map on these cross-sections. The flow near the global piece of the loop Γ outside the d-neighborhood of the saddle defines *the global map* $T_{\rm glo}$ from $\Pi^{\rm out}$ to $\Pi^{\rm in}$. Since the corresponding flight time is bounded, this map is a diffeomorphism and it is well approximated by its Taylor expansion at the point $M^{\rm out}$.

Recall that $H=0$ on $\Pi^{\rm out}$ and $v_2=const\neq 0$. Hence, by (4) and (6), u_2 is a smooth function of (u_1,v_1,u^0,v^0) for points in $\Pi^{\rm out}$. Thus, (u_1,v_1,u^0,v^0) form a good set of coordinates on $\Pi^{\rm out}$. Analogously, (u_2,v_2,u^0,v^0) are the coordinates on $\Pi^{\rm in}$ (here, $u_1=const\neq 0$ and v_1 is found from the condition $H=0$).

Now, we can write the map $T_{\rm glo}: M\mapsto\bar M$ as

$$\begin{cases}
\bar v_2=a_1(v_1-\mu)+b_1u_1+c_1(v^0-v^{0\rm out})+d_1u^0+\dots\\
\bar u_2-u_2^{\rm in}=a_2(v_1-\mu)+b_2u_1+c_2(v^0-v^{0\rm out})+d_2u^0+\dots\\
\bar v^0=a_3(v_1-\mu)+b_3u_1+c_3(v^0-v^{0\rm out})+d_3u^0+\dots\\
\bar u^0-u^{0\rm in}=a_4(v_1-\mu)+b_4u_1+c_4(v^0-v^{0\rm out})+d_4u^0+\dots
\end{cases} \tag{15}$$

where the dots stand for non-linear (quadratic and higher order) terms.

The intersection of $W^{uu}_{\rm loc}$ with $\Pi^{\rm out}$ is $\{v_1=0,u=0\}$, so it follows from (15) that we have

$$\bar v^0=c_3(v^0-v^{0\rm out})$$

on the tangent to $T_{glo}(W^{uu}_{\rm loc} \cap \Pi^{\rm out})$ at $\mu = 0$. The tangent to $W^{see}_{\rm loc}$ is $\bar v^0 = 0$, so the transversality of W^{uu} to W^{see} means that

$$\det c_3 \neq 0.$$

This allows for recasting (15) in the so-called cross-form: $\bar M = T_{glo} M$ if and only if

$$\begin{cases} \bar v_2 = a_1(v_1 - \mu) + b_1 u_1 + c_1 \bar v^0 + d_1 u^0 + \dots \\ \bar u_2 - u_2^{\rm in} = a_2(v_1 - \mu) + b_2 u_1 + c_2 \bar v^0 + d_2 u^0 + \dots \\ v^0 - v^{0\rm out} = a_3(v_1 - \mu) + b_3 u_1 + c_3 \bar v^0 + d_3 u^0 + \dots \\ \bar u^0 - u^{0\rm in} = a_4(v_1 - \mu) + b_4 u_1 + c_4 \bar v^0 + d_4 u^0 + \dots \end{cases} \tag{16}$$

for some new coefficients a, b, c, d, and for some functions of $(v_1 - \mu, u_1, \bar v^0, u^0)$ of at least second order of smallness which are denoted by dots in the right-hand sides of this formula.

When the map is written in the cross-form, it is obvious that the transversality of $T_{glo}(W^u_{\rm loc} \cap \Pi^{\rm out})$ to $W^s_{\rm loc} \cap \Pi^{\rm in}$ at the point $M^{\rm in}$ is equivalent to

$$a_1 \neq 0, \tag{17}$$

and the transversality of $T_{glo}(W^{u1}_{\rm loc} \cap \Pi^{\rm out})$ to $W^s_{\rm loc} \cap \Pi^{\rm in}$ at the point $M^{\rm in}$ is equivalent to

$$b_1 \neq 0. \tag{18}$$

So, our genericity assumptions are (17) and (14) in the case $\lambda_2 < 2\lambda_1$, and (17) and (18) in the case $\lambda_2 > 2\lambda_1$.

We can now introduce the quantities a and A from Theorems 1 and 2:

$$a = -a_1 \tag{19}$$

and

$$A = \begin{cases} -\dfrac{\lambda_2}{\lambda_1} a_1 u_2^{\rm in} & \text{at } \lambda_2 < 2\lambda_1 \\ b_1 d & \text{at } \lambda_2 < 2\lambda_1 \end{cases} \tag{20}$$

Let us now proceed to the evaluation of the *local map* from the cross-sections $\Pi^{\rm in}$ to $\Pi^{\rm out}$ which is defined by the orbits in the d-neighborhood of the saddle O. This is a much less trivial problem because an orbit starting on $\Pi^{\rm in}$ may stay near O for an unboundedly large time before reaching the cross-section $\Pi^{\rm out}$.

The regular method which allows for resolving this difficulty is based upon the study of a specific boundary value problem considered in [7]. Namely, as it follows from [7] for our particular case, if an orbit in a small neighborhood of a saddle starts at $t = 0$ with some point $M_0(u_{10}, u_{20}, u_0^0, v_{10}, v_{20}, v_0^0)$ and reaches a point $M_\tau(u_{1\tau}, u_{2\tau}, u_\tau^0, v_{1\tau}, v_{2\tau}, v_\tau^0)$ at the moment $t = \tau$, then the values of (v_{10}, v_{20}, v_0^0) and $(u_{1\tau}, u_{2\tau}, u_\tau^0)$ are uniquely defined by (u_{10}, u_{20}, u_0^0), $(v_{1\tau}, v_{2\tau}, v_\tau^0)$ and τ. Moreover, such M_0 and M_τ exist for any given $\tau \geq 0$

and small (u_{10}, u_{20}, u_0^0), $(v_{1\tau}, v_{2\tau}, v_\tau^0)$; the corresponding piece of the orbit is found as the unique solution of the following system of integral equations

$$\begin{cases} v_1(t) = e^{-\lambda_1(\tau-t)}v_{1\tau} - \int_t^\tau e^{\lambda_1(t-s)}\,(g_{11}(u(s),v_1(s))v_1(s) + g_{12}(u(s),v_1(s),v_2(s))v_2(s) \\ \qquad\qquad +g_{10}(u(s),v(s))v^0(s)\big)\,ds \\ v_2(t) = e^{-\lambda_2(\tau-t)}v_{2\tau} - \int_t^\tau e^{\lambda_2(t-s)}\,(g_{21}(u(s),v_1(s))v_1(s) + g_{22}(u(s),v_1(s),v_2(s))v_2(s) \\ \qquad\qquad +g_{20}(u(s),v(s))v^0(s)\big)\,ds \\ v^0(t) = e^{-(B^0)^\top(\tau-t)}v_\tau^0 - \int_t^\tau e^{(B^0)^\top(t-s)}\,(g_{01}(u(s),v_1(s))v_1(s) + g_{02}(u(s),v_1(s),v_2(s))v_2(s) \\ \qquad\qquad +g_{00}(u(s),v(s))v^0(s)\big)\,ds \\ u_1(t) = e^{-\lambda_1 t}u_{10} + \int_0^t e^{\lambda_1(s-t)}\,(f_{11}(u_1(s),v(s))u_1(s) + f_{12}(u_1(s),u_2(s),v(s))u_2(s) \\ \qquad\qquad +f_{10}(u(s),v(s))u^0(s)\big)\,ds \\ u_2(t) = e^{-\lambda_2 t}u_{20} + \int_0^t e^{\lambda_2(s-t)}\,(f_{21}(u_1(s),v(s))u_1(s) + f_{22}(u_1(s),u_2(s),v(s))u_2(s) \\ \qquad\qquad +f_{20}(u(s),v(s))u^0(s)\big)\,ds \\ u^0(t) = e^{-B^0 t}u_0^0 + \int_0^t e^{B^0(s-t)}\,(f_{01}(u_1(s),v(s))u_1(s) + f_{02}(u_1(s),u_2(s),v(s))u_2(s) \\ \qquad\qquad +f_{00}(u(s),v(s))u^0(s)\big)\,ds. \end{cases} \tag{21}$$

This system is obtained by integration of (7). According to [7], the solution of (21) on the interval $t \in [0,\tau]$ is found by successive approximations. The first approximation is

$$(u(t) = 0,\ v(t) = 0).$$

Using identities (8), (9), (10) one can see (the detailed computation for a general case can be found in [18,12]) that the second and all the further approximations have the form

$$\begin{aligned} &v_1(t) = e^{-\lambda_1(\tau-t)}v_{1\tau} + O(e^{-\lambda'(\tau-t)}),\ u_1(t) = e^{-\lambda_1 t}u_{10} + O(e^{-\lambda' t}) \\ &v_2(t) = e^{-\lambda_2(\tau-t)}v_{2\tau} + O(e^{-\lambda'(\tau-t)}),\ u_2(t) = e^{-\lambda_2 t}u_{20} + O(e^{-\lambda' t}) \\ &v^0(t) = O(e^{-\lambda'(\tau-t)}), \qquad\qquad u^0(t) = O(e^{-\lambda' t}) \end{aligned} \tag{22}$$

where λ' is some constant such that

$$\lambda' > \min(2\lambda_1, \lambda_2) \tag{23}$$

(note that $\lambda' < \mathrm{Re}\lambda_3$); the $O(\cdot)$-terms in (22) are bounded uniformly, for all successive approximations. Hence, the solution of (21) has the same form.

Note that up to the order $(q-1)$ the derivatives of the successive approximations with respect to the data $\{t, \tau, u_{10}, u_{20}, u_0^0, v_{1\tau}, v_{2\tau}, v_\tau^0\}$ satisfy, uniformly, the estimates obtained by the formal differentiation of (22) (see [18,12]). Therefore, formulas (22) give estimates for the solution of (21) along with the derivatives up to the $(q-1)$-th order.

By (22), the following relation holds for the point M_0 and its time τ shift M_τ:

$$\begin{aligned} &v_{10} = e^{-\lambda_1\tau} v_{1\tau} + O(e^{-\lambda'\tau}),\ u_{1\tau} = e^{-\lambda_1\tau} u_{10} + O(e^{-\lambda'\tau}) \\ &v_{20} = e^{-\lambda_2\tau} v_{2\tau} + O(e^{-\lambda'\tau}),\ u_{2\tau} = e^{-\lambda_2\tau} u_{20} + O(e^{-\lambda'\tau}) \\ &v_0^0 = O(e^{-\lambda'\tau}), \qquad\qquad u_\tau^0 = O(e^{-\lambda'\tau}). \end{aligned} \tag{24}$$

Suppose now that $M_0 \in \Pi^{\text{in}}$ and $M_\tau \in \Pi^{\text{out}}$. It means that $u_{10} = d > 0$, and $v_{2\tau} = d > 0$. Since $H = 0$ at M_0, it follows that

$$v_{10} = -\frac{\lambda_2}{\lambda_1}\frac{u_{20}}{u_{10}} v_{20} - \frac{1}{u_{10}\lambda_1}(v_0^0, B^0 u_0^0) + \ldots \tag{25}$$

where the dots stand for the terms (vanishing at $v_{20} = 0, v_0^0 = 0$) of order higher than two.

Now, it is seen that given any small u_{20}, v_τ^0, u_0^0 and sufficiently large τ the corresponding values of $v_{20}, v_{1\tau}, v_0^0$ and u_τ^0 are defined uniquely and the following estimates hold:

– $\lambda_2 < 2\lambda_1$

$$\begin{aligned} &v_{1\tau} = -\frac{\lambda_2}{\lambda_1}\frac{v_{2\tau}}{u_{10}} u_{20} e^{(\lambda_1-\lambda_2)\tau} + O(e^{(\lambda_1-\lambda')\tau}), \\ &u_{1\tau} = e^{-\lambda_1\tau} u_{10} + O(e^{-\lambda'\tau}), \qquad u_\tau^0 = O(e^{-\lambda'\tau}), \\ &v_{20} = e^{-\lambda_2\tau} v_{2\tau} + O(e^{-\lambda'\tau}), \qquad v_0^0 = O(e^{-\lambda'\tau}); \end{aligned} \tag{26}$$

– $\lambda_2 > 2\lambda_1$

$$\begin{aligned} &v_{1\tau} = O(e^{(\lambda_1-\lambda')\tau}) \\ &u_{1\tau} = e^{-\lambda_1\tau} u_{10} + O(e^{-\lambda'\tau}), \qquad u_\tau^0 = O(e^{-\lambda'\tau}), \\ &v_{20} = O(e^{-\lambda'\tau}), \qquad v_0^0 = O(e^{-\lambda'\tau}). \end{aligned}$$

These formulas define (implicitly) the map T_{loc} from Π^{in} to Π^{out} if we assume u_{20} close to u_2^{in}, u_0^0 close to $u^{0\text{in}}$, v_τ^0 close to $v^{0\text{out}}$ and $u_{10} = v_{2\tau} = d$.

Combining formulas (26) and (16), we arrive to the following formula for the Poincaré map $T = T_{glo} \circ T_{loc} : \Pi^{\text{in}} \to \Pi^{\text{in}}$ (we denote $\nu = \min(\lambda_1, \lambda_2 - \lambda_1)$ and $\mathcal{A} = A$ at $\lambda_2 > 2\lambda_1$ and $\mathcal{A} = A[1 + (u_2 - u_2^{\text{in}})/u_2^{\text{in}}]$ at $\lambda_2 > 2\lambda_1$):

$$\begin{cases} \bar v_2 = a\mu + \mathcal{A} e^{-\nu\tau} + \phi(\bar v^0, \mu) + o(e^{-\nu\tau}), \\ v_2 = o(e^{-\nu\tau}), \qquad v^0 = o(e^{-\nu\tau}) \\ \bar u_2 = u_2^{\text{in}} + \psi(\bar v_2, \bar v^0, \mu) + o(e^{-\nu\tau}), \\ \bar u^0 = u^{0\text{in}} + \psi^0(\bar v_2, \bar v^0, \mu) + o(e^{-\nu\tau}), \end{cases} \tag{27}$$

where ϕ, ψ, ψ^0 are some smooth functions vanishing at zero:

$$\bar u_2 = u_2^{\text{in}} + \psi(\bar v_2, \bar v^0, \mu), \qquad \bar u^0 = u^{0\text{in}} + \psi^0(\bar v_2, \bar v^0, \mu) \tag{28}$$

is the equation of the surface w^{u*} equal to $T_{glo}(W^u_{\text{loc}} \cap \Pi^{\text{out}})$ at $\lambda_2 < 2\lambda_1$ and to $T_{glo}(W^{u1}_{\text{loc}} \cap \Pi^{\text{out}})$ at $\lambda_2 > 2\lambda_1$; the subset of this surface given by the equation

$$\bar v_2 = a\mu + \phi(\bar v^0, \mu) \tag{29}$$

is $w^{uu} = T_{glo}(W^{uu}_{\text{loc}} \cap \Pi^{\text{out}})$.

Since $\mathcal{A} \neq 0$ (recall that $u_2 - u_2^{\text{in}}$ is small on Π^{in}), it follows that the first equation of (27) can be resolved with respect to τ, provided

$$A(\bar v_2 - a\mu - \phi(\bar v^0, \mu)) > 0.$$

If we make an additional change of coordinates on Π^{in}:

$$\begin{array}{l} u_{2,new} = u_2 - u_2^{\text{in}} - \psi(v_2, v^0, \mu), \\ u^0_{new} = u^0 - u^{0\text{in}} - \psi^0(v_2, v^0, \mu), \\ v_{2,new} = v_2 - \phi(v^0, \mu), \end{array} \tag{30}$$

so that equations of w^{u*} and w^{uu} become, respectively,

$$w^{u*}: \quad (u_2, u^0) = 0 \tag{31}$$

and

$$w^{uu}: \quad (u_2, u^0) = 0, \quad v_2 = a\mu, \tag{32}$$

then, after resolving (27) with respect to τ, the Poincaré map T can be written in the following form

$$(\bar u_2, \bar u^0, v_2, v^0) = \xi(u_2, u^0, \bar v_2, \bar v^0) \tag{33}$$

where ξ is a smooth function defined at

$$A(\bar v_2 - a\mu) > 0 \tag{34}$$

and vanishing at $\bar v_2 = a\mu$ along with the first derivatives, so that

$$\xi = o(\bar v_2 - a\mu). \tag{35}$$

If we assume $\xi = 0$ at $A(\bar{v}_2 - a\mu) < 0$, then the right-hand side of (33) will define a contracting map. Its unique fixed point $M^*(u_2^*, v_2^*, u^{0*}, v^{0*})$ will be a fixed point of the Poincaré map T if and only if v_2^* satisfies (34). By (35),

$$v_2^* = o(v_2^* - a\mu), \tag{36}$$

so it is obvious now that the map T has a fixed point if and only if $Aa\mu < 0$.

The fixed point of the Poincaré map corresponds to the periodic orbit L_μ. By (36), $v_2^* \to 0$ as $\mu \to 0$. By (33), it follows that $(u_2^*, v_2^*, u^{0*}, v^{0*}) \to 0$ as $\mu \to 0$, i.e. the periodic orbit merges into the homoclinic loop Γ_0 at $\mu = 0$.

Take some $K > 0$ and let us call as a vertical surface a surface of the kind $(u_2, u^0) = \eta(v_2, v^0)$ with $\|\eta'\| \le K$ and let a horizontal surface be a surface of the kind $(v_2, v^0) = \nu(u_2, u^0)$ with $\|\nu'\| \le K$. It is immediately seen from (33)-(35) that for every $K > 0$, if the range of μ and v_2 is sufficiently small, the preimage of any horizontal surface which intersects the region $A(v_2 - a\mu) > 0$ is a horizontal surface again, and the image of any vertical surface is a piece of a vertical surface (this piece is bounded by w^{uu} and lies in the region $A(v_2 - a\mu) > 0$). Moreover, when restricted to a vertical surface the map T is expanding and it is contracting on horizontal surfaces.

Thus, the map T has a hyperbolic structure and, in particular, its fixed point is a saddle (so L_μ is a saddle periodic orbit) whose stable manifold is a horizontal surface and the unstable manifold is a piece of a vertical surface. Due to the hyperbolicity, all the orbits of the map T must leave Π^{in} after a number of iterations (forward or backward), except for the fixed point. For the flow itself, this means that the only orbits which may stay in a small neighborhood U of the loop are the periodic orbit L_μ and, possibly, some orbits in $W^s(O)$ or $W^u(O)$ (such orbits correspond to finite, at least from one side, orbits of the Poincaré map T).

The orbits from $W^u(O)$ or $W^s(O)$ correspond to the orbits of the map T starting on $w^u = T_{glo}(W^u_{\text{loc}} \cap \Pi^{\text{out}})$ or, respectively, ending on $w^s = W^s_{\text{loc}} \cap \Pi^{\text{in}} = \{v_2 = 0, v^0 = 0\}$. If such an orbit is infinite to the right, it must start with a point on w^u and tend to the fixed point M^*. Thus, it must belong to the stable manifold of M^*, i.e. the starting point on w^u is defined uniquely as the intersection of $w^s(M^*) \cap w^u$ (this intersection is unique because $w^s(M^*)$ is a horizontal surface and w^u is vertical, by our assumption of the transversality of w^u and w^s). This gives us a unique heteroclinic orbit $C_{1\mu}$ which is α-limit to O and ω-limit to L_μ.

The rest are homoclinic loops and the heteroclinic orbit $C_{2\mu}$ which is α-limit to L_μ and ω-limit to O. We start with homoclinic loops. They correspond to the intersection of w^u with w^s (the original loop Γ) and with its preimages $w^s_k = T^{-k} w^s$. When exists, each of these preimages is a horizontal surface which, hence, has a unique intersection point with w^u and this intersection corresponds to the homoclinic loop $\Gamma_{k\mu}$. Thus, the problem of existence of homoclinic loops is reduced to the following question: until which k the surfaces w^s_k intersect the region $A(v_2 - a\mu) > 0$? At $Aa\mu \ge 0$, the sur-

face w^s itself does not lie in this region so it has no preimages. Therefore, no homoclinic loops $\Gamma_{k\mu}$ exist with $k \geq 1$ (heteroclinic orbits cannot exists either because there is no periodic orbit at these μ). When $Aa\mu < 0$, the surface w^s lie in $A(v_2 - a\mu) > 0$. Hence, it has a preimage w_1^s. By (33),(35), we have $v_2 = o(\mu)$ on w_1^s, therefore $A(v_2 - a\mu) > 0$ on w_1^s, so it has a preimage as well, and so on: we obtain the infinite sequence of preimages w_k^s for all of which $v_2 = o(\mu)$ uniformly. Thus we have proved the existence of homoclinic loops $\Gamma_{k\mu}$ at $Aa\mu < 0$. Since the horizontal surfaces w_k^s stay all in a bounded region they must accumulate to the stable manifold of the saddle fixed point M^*. Therefore, they must intersect the unstable manifold of M^* which gives us the existence of the heteroclinic orbit $C_{2\mu}$ which is α-limit to L_μ and ω-limit to O (this orbit is unique because w^s can have no more than one intersection with $W^u(M^*)$ since the latter is a piece of a vertical surface). This finishes the proof of theorem 1.

To prove theorem 2, note that in a small neighborhood of O there is no orbit which starts in a small neighborhood of a point in $W^s \backslash W^{ss}$ with $\{H = 0\}$ and comes in a small neighborhood of any point in $W^u \backslash W^{uu}$. Indeed, for such an orbit we would have $v_{1\tau} \neq 0$ and $u_{10} \neq 0$ in formula (24), and this makes it clearly impossible to have $H(M_0) = 0$ or $H(M_\tau) = 0$ at sufficiently large τ (recall that the large flight time τ corresponds to the orbits starting close to the stable invariant manifold of O).

Therefore, any orbit which stays in a small neighborhood U of the homoclinic bunch $\Gamma \cup \Gamma_+^1 \cup \ldots \cup \Gamma_+^{m_+} \cup \Gamma_-^1 \cup \ldots \cup \Gamma_-^{m_-} \cup O$ in the level $\{H = 0\}$ and which starts close to a loop $\Gamma_\pm^i$ must enter a small neighborhood of Γ (and stay there after that) immediately after one passage near O. Thus, except for the orbits which stay all the time in a small neighborhood of Γ, the system may have in U only such orbits which start in $W_{\rm loc}^u(O)$, make one round near one of the loops $\Gamma_\pm^i$ and then enter a small neighborhood of Γ. To stay there, these orbits must either come into $W_{\rm loc}^s \cap \Pi^{\rm in}$ after a number of rounds near Γ, or they must belong to the stable manifold of the periodic orbit L_μ which exists at $Aa\mu < 0$. So, to prove the theorem we must, for every loop Γ_γ^i ($\gamma = \pm$), take a small piece of $W_{\rm loc}^u(O)$ near this loop, continue it by the orbits of the flow close to the loop back to a small neighborhood of O, then trace how it goes to the loop Γ, make one round near Γ and examine how the obtained surface intersects (on the cross-section $\Pi^{\rm in}$) the surface $w^s = W_{\rm loc}^s \cap \Pi^{\rm in}$ (this intersection will correspond to a double loop $(i\gamma)0$) and, at $Aa\mu < 0$, the surfaces $w_k^s = T^{-k} w^s$ (these intersections will correspond to the loops $(i\gamma)0^k$) and the stable manifold $w^s(M^*)$ of the saddle fixed point of T (this intersection will correspond to the heteroclinic orbit $(i\gamma)0^\infty$).

Let $\Pi_{i\pm}^{\rm in}$ be small cross-sections to the local stable manifold, intersecting the loops $\Gamma_\pm^i$, respectively. We may assume that $u_1 = d > 0$ on $\Pi_{i+}^{\rm in}$ and $u_1 = -d < 0$ on $\Pi_{i-}^{\rm in}$. A piece of $W_{\rm loc}^u$ mapped by the flow near a loop Γ_γ^i on the cross-section $\Pi_{i\gamma}^{\rm in}$ is a surface transverse to $W_{\rm loc}^s$. The image of this surface by the local map on the cross-section $\Pi^{\rm out}$ to the loop Γ is found by formulas

(24) where one should put $u_{10} = d > 0$ at $\gamma = +$ and put $u_{10} = -d < 0$ at $\gamma = -$ (recall that $v_{2\tau} = d > 0$ on Π^{out}). Thus, this image is a surface tangent to $W^u_{\text{loc}} \cap \Pi^{\text{out}}$ in the case $\lambda_2 < 2\lambda_1$ or to $W^{u1}_{\text{loc}} \cap \Pi^{\text{out}}$ in the case $\lambda_2 > 2\lambda_1$. In both cases this surface is bounded by $W^{uu}_{\text{loc}} \cap \Pi^{\text{out}}$.

When applying the global map (16) to this surface we will obtain a vertical surface (in the coordinates given by (30)) adjoining to w^{uu} from the side $\gamma A(v_2 - a\mu) > 0$. It is seen now immediately that this surface has an intersection (and this intersection is transverse and unique) with w^s and with any horizontal curve $o(\mu)$-close to w^s (at $Aa\mu < 0$ such are the preimages w^s_k of w^s and their limit $w^s(M^*)$; see the proof of theorem 1) if and only if $\gamma Aa\mu < 0$. This is in a complete correspondence with the statement of theorem 2. End of the proof.

Theorem 2 treats the case of a finite number of loops $\Gamma^i_{\pm}$, but it can be easily generalized to the case of an infinite set of loops. Namely, let a number of saddle periodic orbits $L_1, \ldots, L_m$ exists in the level $\{H = 0\}$ at $\mu = 0$ (hence, at all small μ). Suppose the unstable manifold of L_i intersects the stable manifold of L_j transversely at some number $m_{ij} \geq 0$ of heteroclinic (homoclinic at $i = j$) orbits C_{ijs} ($s = 1, \ldots, m_{ij}$ at $m_{ij} \geq 1$). Then (see [7,20]), one can take a sufficiently small neighborhood V of $L_1 \cup \ldots \cup L_m \cup_{ijs} C_{ijs}$ in the level $\{H = 0\}$ such that the set N of all orbits staying in V entirely will be a hyperbolic set topologically conjugate to a subshift of finite type, described by the following transition graph G (oriented): it has m vertices denoted as $L_1, \ldots, L_m$ and, for every $i = 1, \ldots, m$, from the vertex L_i one edge, denoted also as L_i, goes to the same vertex, plus m_{ij} edges denoted as $L_i^{\bar{k}} C_{ijs} L_j^{\bar{k}}$ ($s = 1, \ldots, m_{ij}$) go to the vertex L_j, for every $j = 1 = 1, \ldots, m$; here $\bar{k}$ is a sufficiently large integer. In other words, for every infinite oriented path in graph G, in V there exists an orbit whose natural coding is read from the consecutive edges in this path, and this correspondence between the paths in the graph and the orbits of N is one-to-one and continuous. Every orbit of N has local stable and unstable manifolds the size of which is bounded away from zero. If the codings of two forward semiorbits are close, then their stable manifolds are close as well; also, if the codings of two backward semiorbits are close, then their unstable manifolds are close.

Let $W^u(O)$ intersect transversely the stable manifolds of periodic orbits L_i at $m_{0i} \geq 0$ heteroclinic orbits C_{0is}, $s = 1, \ldots, m_{0i}$ at $m_{0i} \geq 1$, $i = 1, \ldots, m$, and let $W^s(O)$ intersect transversely the unstable manifolds of periodic orbits L_i at $m_{i0} \geq 0$ heteroclinic orbits C_{i0s}, $s = 1, \ldots, m_{i0}$ at $m_{i0} \geq 1$. Then, by λ-lemma, m_{0i} pieces of $W^u(O)$ will come sufficiently close to the local unstable manifold of L_i; hence, each of them will have one point of transverse intersection with the stable manifold of every orbit of N close to L_i. Analogously, m_{i0} pieces of $W^s(O)$ will come sufficiently close to the local stable manifold of L_i, so each of these pieces will have one point of transverse intersection with the unstable manifold of every orbit of N close to L_i. Thus, if we enlarge the neighborhood V by adding to it a small neighborhood of O and

the heteroclinic orbits C_{0is} ($s \le m_{0i}$) and C_{i0s} ($s \le m_{i0}$, $i = 1, \dots, m$) in the level $\{H = 0\}$, then in the new V there will exist a set $\tilde{N} \supseteq N$ of orbits for which the natural coding will give a one-to-one continuous correspondence with the set of the oriented paths (infinite, or starting at O^u, or ending in O^s) in the graph $\tilde{G}$ obtained from G by adding a pair of vertices O^s and O^u with the edges $C_{0is}L_i^{\bar{k}}$ ($s \le m_{0i}$) aiming from O^u to L_i and $L_i^{\bar{k}}C_{0is}$ ($s \le m_{i0}$) aiming from L_i to O^s, $i = 1, \dots, m$. By construction, the paths starting with O^u and ending at O^s correspond to homoclinic loops, and if the graph G is nontrivial, the set of these loops will be infinite, of course.

When all the heteroclinic orbits C_{0is} and C_{i0s} are in general position, i.e. they do not lie in strong unstable or, respectively, strong stable manifolds W^{uu} and W^{ss} of O, there are no other orbits lying entirely in V except for O and those from the set $\tilde{N}$ described above. This follows from the fact we established while proving theorem 2 that in a neighborhood of O there can be no orbit which would lie in $\{H = 0\}$ and pass from a small neighborhood of a point in $W^s \backslash W^{ss}$ with $\{H = 0\}$ to a small neighborhood of any point in $W^u \backslash W^{uu}$ - hence, every positive or negative semiorbit in V which comes close to O must enter $W^s_{\rm loc}(O)$ or, respectively, $W^u_{\rm loc}(O)$, so it belongs to the set $\tilde{N}$ indeed.

So, we assume that C_{0is} and C_{i0s} are in general position. Moreover, we divide the orbits C_{i0s} into two groups: those lying in W^{s+} and those lying in W^{s-}. Accordingly, we change notations denoting these heteroclinics as C_{i0s+} ($s \le m_{i0+}$) and C_{i0s-} ($s \le m_{i0-}$) where m_{i0+} and m_{i0-} are the number of the orbits in W^{s+} and the number of the orbits in W^{s-} respectively, so that $m_{i0+} + m_{i0-} = m_{i0}$. We also change the graph $\tilde{G}$ by splitting the vertex O^s into two: O^{s+} and O^{s-}, so that the edges corresponding to the orbits C_{i0s+} end at O^{s+} and those corresponding to C_{i0s-} end at O^{s-}.

Let $\Pi^{\rm in}_{is\pm}$ be small cross-sections to the local stable manifold, intersecting the orbits $C_{i0s\pm}$, respectively. We may assume that $u_1 = d > 0$ on $\Pi^{\rm in}_{is+}$ and $u_1 = -d < 0$ on $\Pi^{\rm in}_{is-}$. A piece of $W^u_{\rm loc}(L_i)$ mapped by the flow near an orbit $C_{i0s\gamma}$ ($\gamma = \pm$) on the cross-section $\Pi^{\rm in}_{is\gamma}$ is a surface transverse to $W^s_{\rm loc}(O)$. Since the local unstable manifolds of the backward orbits in $\tilde{N}$ depend continuously on their coding, local unstable manifolds of all backward orbits in $\tilde{N}$ whose coding start with a sufficiently long sequence of L_i's lie close to $W^s_{\rm loc}(L_i)$ (at least in C^1-sense). Therefore, if we took the value of $\bar{k}$ sufficiently large when constructing the set $\tilde{N}$, we will have for every path g in the graph $\tilde{G}$ which ends with the edge $C_{i0s\gamma}$ that the unstable manifold of the corresponding backward semiorbit intersects $\Pi^{\rm in}_{is\gamma}$ at a surface w^s_g transverse to $W^s_{\rm loc}(O)$ and the sizes of these surfaces are bounded away from zero, as well as the angles they form with $W^s_{\rm loc}(O)$.

Let us now assume that at $\mu = 0$ there exists a homoclinic orbit Γ undergoing the orbit-flip bifurcation and the genericity assumptions of theorem 1 hold. We can now apply the arguments of theorem 2 to the surfaces w^s_g, uniformly to all of them. This will give that the images of these surfaces by

the local map on the cross-section $\Pi^{\rm out}$ to the loop Γ are some surfaces, whose size is bounded away from zero, confined all in a small angle around $W^u_{\rm loc} \cap \Pi^{\rm out}$ in the case $\lambda_2 < 2\lambda_1$ or around $W^{ul}_{\rm loc} \cap \Pi^{\rm out}$ in the case $\lambda_2 > 2\lambda_1$. In both cases the surfaces are bounded by $W^{uu}_{\rm loc} \cap \Pi^{\rm out}$. All the surfaces coming from $\Pi^{\rm in}_{is+}$ adjoin to $W^{uu}_{\rm loc} \cap \Pi^{\rm out}$ from one side and the surfaces coming from $\Pi^{\rm in}_{is-}$ adjoin to $W^{uu}_{\rm loc} \cap \Pi^{\rm out}$ from the other side, exactly by the same rule as in theorem 2. Thus, exactly like in theorem 2, we arrive at the following statement.

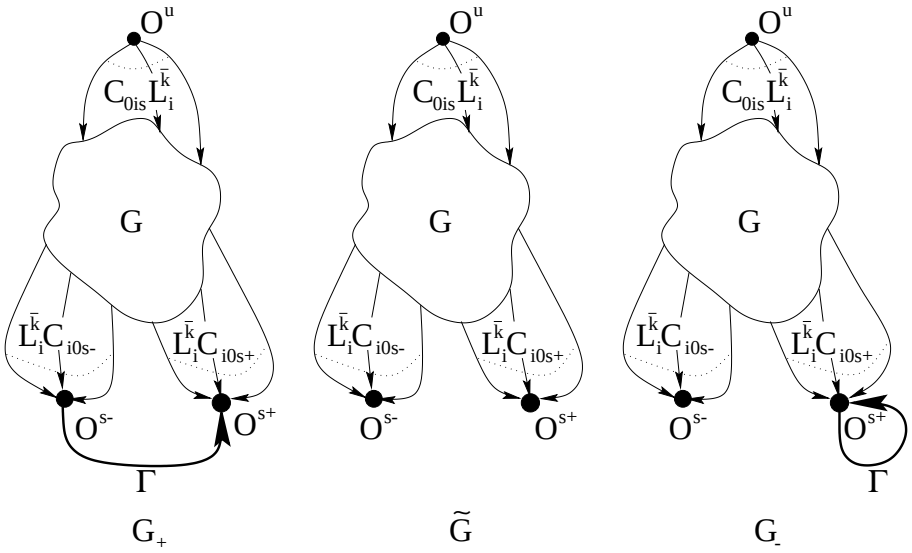

Fig. 5. The graphs G_+ and G_- are obtained from $\tilde{G}$ by adding one edge labelled Γ which ends at O^{s+} and starts at O^{s-} or O^{s+}, respectively.

Theorem 3. *Let U be the union of the neighborhood V of the set $\tilde{N}$ with a small neighborhood of Γ in $\{H = 0\}$. Then the set of all orbits lying in U entirely is (excluding O and Γ) in one-to-one continuous correspondence with the oriented paths in the graph $\tilde{G}$ at $\mu = 0$, G_+ at $Aa\mu > 0$ and G_- at $Aa\mu < 0$ where G_+ and G_- are obtained from $\tilde{G}$ by adding one more edge Γ which starts with O^{s-} or O^{s+}, respectively, and ends at O^{s+} in both cases (Fig.5). The homoclinic loops correspond to the paths starting with O^u and ending at one of the vertices $O^{s\pm}$.*

3 Superhomoclinic orbits

Let us now consider in more detail the behavior of orbits in a small neighborhood of the homoclinic loop Γ at the moment of the orbit-flip bifurcation (i.e. at $\mu = 0$). The problem reduces to the study of the Poincaré map T on the cross-section Π^{in}. By (33)-(35), the map T is written in the following form

$$(\bar{u}_2, \bar{u}^0, v_2, v^0) = \xi(u_2, u^0, \bar{v}_2, \bar{v}^0) = o(\bar{v}_2) \tag{37}$$

where ξ is a smooth function defined at

$$A\bar{v}_2 > 0 \tag{38}$$

and vanishing at $\bar{v}_2 = 0$ along with the first derivatives. If we define the function ξ at $A\bar{v}_2 \le 0$ as $\xi = 0$, then the right-hand side of (37) will be a smooth function defined for all small $u_2, u^0, \bar{v}_2, \bar{v}^0$, whose first derivatives will be all small. Hence, theorem 4.4 of [15] is applied which gives the existence of a smooth attracting invariant manifold $\tilde{w}$ for the map T. Namely, this manifold $\tilde{w}$ has the form

$$(u_2, u^0) = \tilde{\eta}(v_2, v^0) \tag{39}$$

for some smooth function $\tilde{\eta}$ (the invariance of this manifold implies that $\tilde{\eta}$ vanishes at $\bar{v}_2 = 0$ along with the first derivatives), and every forward semiorbit of T which never leaves Π^{in} must tend uniformly to $\tilde{w}$. Hence, every infinite backward semiorbit of T must lie in $\tilde{w}$.

Note that it is obvious from (37) that on $\tilde{w}$ the map T^{-1} is defined and strongly contracting everywhere in the region (34). We will show that the orbits of the flow which start on $\tilde{w}$ with $v_2 \le 0$ do not come to the cross-section Π^{out} after passing near the saddle O, so they do not return to Π^{in}. This means that the domain of the Poincaré map T on $\tilde{w}$ lies in the region $v_2 > 0$, i.e. the contracting map T^{-1} maps the region $Av_2 > 0$ inside the region $v_2 > 0$. Hence, at $A < 0$ the backward semiorbit of every point in $\tilde{w}$ leaves Π^{in} with the iterations of T^{-1}, whereas at $A > 0$ for every point in $\tilde{w}$ with positive v_2 its backward semiorbit stays in Π^{in}. Since T^{-1} is contracting, all infinite semiorbits must tend to the fixed point in the origin in Π^{in}. Thus, we have that the manifold

$$\tilde{w}^u : (u_2, u^0) = \tilde{\eta}(v_2, v^0), v_2 > 0 \tag{40}$$

is the *unstable manifold* of the origin in Π^{in} at $A > 0$. Since this point is the intersection point of Γ with Π^{in}, it follows that the orbits of the flow which pass through the points of w^u have the homoclinic loop Γ as the α-limit set. This gives us the following result:

Lemma 1. *Let $A > 0$ for the homoclinic loop Γ at the moment of the orbit-flip bifurcation. Then, the unstable set of Γ (i.e. the set of all orbits*

which tend to Γ as $t \to -\infty$) is non-empty and it is a smooth n-dimensional manifold $W^u(\Gamma)$ with the boundary $W^{uu}(O)$ which is tangent at the points of Γ to W^u if $\lambda_2 < 2\lambda_1$ and to W^{u1} if $\lambda_2 > 2\lambda_1$. All the orbits in $\{H = 0\}$ which do not belong to $W^u(\Gamma)$, W^s or W^u leave a small neighborhood of Γ both as $t \to +\infty$ and $t \to -\infty$.

To prove this statement it remains to show that the orbits of the flow which start on $\tilde{w}$ with $v_2 \le 0$ do not come to Π^{out}. Recall that we assume the transversality of the manifolds $W^{see}(O)$ and $W^{uu}(O)$ at the points of Γ, which is equivalent to the existence of an $(n+2)$-dimensional repelling smooth invariant manifold $W^{see}(\Gamma)$ which contains Γ and $W^s_{\text{loc}}(O)$ and which is transverse to W^{uu} at O [17,16] (it is tangent to $v^0 = 0$ at O, in fact).

The intersection of $W^{see}(\Gamma)$ with Π^{in} is a surface

$$w^{see} : v^0 = \varphi(u_2, u^0, v_2)$$

with some smooth function φ vanishing at $v_2 = 0$. By construction, w^{see} is invariant with respect to T. Since the derivatives of the function η in (39) are small at small v_2, it follows that w^{see} intersects the invariant manifold $\tilde{w}$ along a smooth invariant curve

$$w^* : (v^0, u_2, u^0) = \zeta(v_2)$$

where $\zeta(0) = 0$. The orbits which start on w^* lie in the invariant manifold $W^{see}(\Gamma)$; since the latter is transverse to W^{uu}, it follows that $v^0 = O(v_1, v_2)$ for every orbit starting with w^*, all the time this orbit lies in a neighborhood of O (moreover, W^{see} is tangent to $v^0 = 0$ at O, so we also have that $\|v^0\| \ll d$). Therefore, the evolution of the v_2-coordinate on this orbit is given by the equation of the form

$$\dot{v}_2 = \lambda_2 v_2 + o(v_1, v_2)$$

(see (7). By (24), the ratio u_2/u_1 remains uniformly bounded for this orbit (since $\lambda_2 > \lambda_1$ and $u_1 = d \neq 0$ initially). Hence, since the orbit lies in $\{H = 0\}$, it follows that $v_1 = O(v_2)$ and we have

$$\dot{v}_2 = \lambda_2 v_2 + o(v_2).$$

It is now obvious that the orbits which start on w^* with nonpositive v_2 can never enter the region of positive v_2 so they leave the d-neighborhood of O through the cross-section $v_2 = -d$ (the orbit Γ which pass through the point $v_2 = 0$ on w^* tends to O).

Now, take any point M on $\tilde{w}$ with $v_2 \le 0$ and let M^* be the point of intersection of the surface $\{v_2 = const\}$ through the point M with w^*. This surface is transverse to w^{see}. Since the cone $\|u, v_1, v_2\| \le K\|v^0\|$ is, at every K, invariant with respect to the forward flow linearized at the point O, it follows that the tangents to every surface obtained by the forward shift by the local flow near O of a surface transverse to w^{see} belong all to such cone

with a sufficiently large K, provided the size d of the neighborhood of O under consideration is small enough. Thus, the forward time t shift M_t of the point M will remain in such a cone with the vertex at the time t shift M_t^* of the point M^* which makes it impossible for M_t to belong to $\Pi^{\rm out}$ (in that case both M_t and M_t^* would have $\|v^0\| \ll d$ but the difference in v_2 would be of order d which would contradict the invariant cone property). This proves the claim.

The invariant manifold $W^u(\Gamma)$ is n-dimensional and lies in the level $\{H = 0\}$. Hence, it may have orbits of the transverse (in this level) intersection with $W^s(O)$. We call such orbits *superhomoclinic*. Let S be a superhomoclinic orbit of transverse intersection of $W^u(\Gamma)$ with $W^s(O)$. Assume that S enters O at $t = +\infty$ along the leading direction, i.e. it is tangent to the u_1-axis. Moreover, we assume that S adjoins O from the side of positive u_1, i.e. $S \subseteq W^u(\Gamma) \cap W^{s+}$ (as we will see the case $S \subseteq W^u(\Gamma) \cap W^{s-}$ is trivial). Let U be a small neighborhood of $\Gamma \cup S \cup O$. It is a ball (around O) with two handles around Γ and S. We can therefore consider a natural code for the orbits in U describing the sequence of the handles visited by the orbits. Note that the codings of the orbits in $W^u(O)$ are finite to the left and the codings of the orbits in $W^s(O)$ are finite to the right, so the codings of homoclinic loops are finite to both sides; the coding of O is empty.

Let Ω be the set of sequences of symbols S and Γ constructed by the following rule: for some positive integer $\bar{k}$ take all infinite or starting with Γ and infinite to the right sequences obtained by repeated concatenation of subsequences Γ and $S\Gamma^{\bar{k}}$ in an arbitrary order; then change the infinite sequence composed of Γ's only to the one-symbol sequence $\{\Gamma\}$ and, for every other sequence which ends by the infinite string of Γ's, omit this string; the set thus obtained plus the empty sequence is the set Ω.

Theorem 4. *There exists a sufficiently large $\bar{k}$ and a small neighborhood U of $\Gamma \cup S \cup O$ such that the set of all orbits lying entirely in U is in one-to-one correspondence (provided by natural coding) with Ω.*

Proof. The intersection of $W^u(\Gamma)$ with the cross-section $\Pi^{\rm in}$ is the invariant manifold $\tilde{w}^u$ of the Poincaré map T. The manifold $\tilde{w}^u$ is given by (40) but we will change coordinates on $\Pi^{\rm in}$ such that it would have the equation

$$u = 0,\ v_2 > 0; \tag{41}$$

since the function $\tilde{\eta}$ in (40) vanishes at zero along with its derivatives, this coordinate transformation would not change the formula (37), nor it would change the formula (16) for the map $T_{glo} : \Pi^{\rm out} \to \Pi^{\rm in}$.

Let $P(0, v_P) \in \tilde{w}^u$ be a point of intersection of the superhomoclinic orbit S with $\Pi^{\rm in}$. By assumption, this orbit belongs to the stable manifold of O, hence it must eventually come to $W^s_{\rm loc}(O)$. Moreover, this orbit lies in W^{s+}. Hence, it must intersect the cross-section $\{u_1 = d\}$ at some point $Q(u_Q, 0) \in W^s_{\rm loc}(O)$. Let $\tilde{\Pi}^{\rm in}$ be a piece of the cross-section $\{u_1 = d\}$ around

Q in the level $\{H = 0\}$. The flow near S defines a map T_S from a small neighborhood of P on Π^{in} into $\tilde{\Pi}^{\text{in}}$, so that $Q = T_S P$.

The map T_S corresponds to a finite flight time, so its derivatives are bounded and it is well approximated by its linearization at the point P, like the map T_{glo} near Γ. We can write T_S in the following form

$$\begin{cases} \tilde{v} = \tilde{a}(v - v_P) + \tilde{b}u + \dots \\ \tilde{u} - u_Q = \tilde{c}(v - v_P) + \tilde{d}u + \dots \end{cases} \tag{42}$$

where the dots stand for non-linear (quadratic and higher order) terms; $(\tilde{u}, \tilde{v})$ denote coordinates on $\tilde{\Pi}^{\text{in}}$. Note that by assumption of the transversality of $W^u(\Gamma)$ to $W^s(O)$ the image of a small piece of surface $u = 0$ around the point P by the map T_S is a surface transverse to $v = 0$ in $\tilde{\Pi}^{\text{in}}$. It means that $\tilde{a} \neq 0$ in (42).

The map from $\tilde{\Pi}^{\text{in}}$ to Π^{out} is given by formulas (24) where one should put $u_{10} = d > 0$ and $v_{2\tau} = d > 0$. Note that the flight time τ must be taken sufficiently large because $\tilde{\Pi}^{\text{in}}$ is a small neighborhood of the point Q which lies in $W^s_{\text{loc}}(O)$ and whose forward orbit stays, therefore, infinitely long time in the d-neighborhood of O. Now, combining formulas (42),(24) and (16), one can see that the map $T_{glo}T_{loc}T_S$ by the flow from a small neighborhood of P in Π^{in} close to the superhomoclinic orbit S and then close to Γ back to Π^{in} is given by the formula

$$(\bar{u}_2, \bar{u}^0, v_2 - \nu_2^s, v^0 - \nu^{0s}) = \tilde{\xi}(u_2, u^0, \bar{v}_2, \bar{v}^0) = o(\bar{v}_2) \tag{43}$$

where $\tilde{\xi}$ is a smooth function defined at sufficiently small u and sufficiently small positive[2] $\bar{v}_2$ and vanishing at $\bar{v}_2 = 0$ along with the first derivatives, and $u = \nu^s(v)$ is the preimage of $W^s_{\text{loc}} \cap \tilde{\Pi}^{\text{in}}$ on Π^{in}; by construction, $0 = \nu^s(v_P)$.

Note that we cannot control the range of $\bar{v}_2$ for which the function $\tilde{\xi}$ is defined (we only know that it is defined at sufficiently small positive $\bar{v}_2$ which corresponds to sufficiently large time τ of the flight from $\tilde{\Pi}^{\text{in}}$ to Π^{out}). In particular, the value of v_{2P} can be out of the domain of $\tilde{\xi}$. However, it is easy to see from (43) and (37) that for a sufficiently large $\bar{k}$ the map $\tilde{T} = T^{\bar{k}-1}T_{glo}T_{loc}T_S$ from a small neighborhood of P is still written in the form (43) where the function $\tilde{\xi}$ is defined for $\bar{v}_2 \in (0, \delta]$ with some $\delta > v_{2P}$ and the range of the map $(u, \bar{v}) \mapsto (\bar{u}, v)$ defined by formula (43) now lies inside its domain (the domain of $\tilde{\xi}$).

If we define the functions $\tilde{\xi}$ and ξ in formulas (43) and (37), respectively, as zero at $\bar{v}_2 = 0$, we obtain a rectangular domain in Π^{in} where a pair of maps T and $\tilde{T}$ are defined, for both of which the corresponding cross-maps $(u, \bar{v}) \mapsto (\bar{u}, v)$ take this domain into itself and they are both strongly

[2] Note that if $S \in W^{s-}$, we would have $u_{10} = -d < 0$ in (24) which would give $\bar{v}_2 < 0$ in (43). Thus, the orbits starting close to P would return to that part of Π^{in} where further iterations of T or $\tilde{T}$ are not defined. Hence, in that case, no orbits other than S, Γ and O can lie in U entirely.

contracting. Thus, the lemma [7] on a saddle fixed point of a sequence of saddle operators in the product of Banach spaces is applied here which gives that for every sequence $\{\sigma_i\}_{i=-\infty}^{+\infty}$ of symbols 0 and 1 there exists a unique sequence of points M_i such that $M_{i+1} = TM_i$ if $\sigma_i = 0$ and $M_{i+1} = \tilde{T}M_i$ if $\sigma_i = 1$. Moreover, the points M_i depend continuously on the corresponding sequences $\{\sigma_i\}_{i=-\infty}^{+\infty}$ and each of these points has a stable manifold which is a horizontal surface (i.e. a surface of the kind $v = \nu(u)$ where the derivative of ξ is sufficiently small). Every such surface has a unique point of the transverse intersection with the vertical surface $w^u = T_{glo}(W^u_{\rm loc} \cap \Pi^{\rm out})$. Thus, for every infinite to the right sequence $\{\sigma_i\}_{i=0}^{+\infty}$ there exists a unique sequence of points M_i such that $M_0 \in w^u$ and $M_{i+1} = TM_i$ if $\sigma_i = 0$ and $M_{i+1} = \tilde{T}M_i$ if $\sigma_i = 1$.

The obtained sequences $\{M_i\}$ correspond to the trajectories of the original maps T and $\tilde{T}$ if and only if the coordinate v_2 is not zero for every point M_i in the sequence. If $\bar{v}_2 = 0$ at some point M_{i+1}, it means that the corresponding values of $\tilde{\xi}$ or ξ are zero in, respectively, (43) or (37). Hence, M_{i+1} is the origin in $\Pi^{\rm in}$, i.e. $M_{i+1} = \Gamma \cap \Pi^{\rm in}$, and either $M_{i+1} = \tilde{T}M_i$ - in this case $M_i \in (T_{glo}T_{loc}T_S)^{-1}(W^s_{\rm loc} \cap \tilde{\Pi}^{\rm in})$, or $M_{i+1} = TM_i$ - in this case $v_2 = 0$ at the point M_i which means that $M_i = \Gamma \cap \Pi^{\rm in}$ as well. Thus, we have that either $\{\sigma_i\}$ consists of all 0's, so all the points of the corresponding sequence $\{M_i\}$ are the same fixed point $\Gamma \cap \Pi^{\rm in}$ of T, or all points M_i have $v_2 \neq 0$, or there is a point $M_i \in (T_{glo}T_{loc}T_S)^{-1}(W^s_{\rm loc} \cap \tilde{\Pi}^{\rm in})$ for which all the previous points have nonzero v_2 and $\tilde{T}M_i = M_{i+1} = \Gamma \cap \Pi^{\rm in}$ which means that $\sigma_i = 1$ and all the further symbols are 0's. Vice versa, if the sequence $\{\sigma_i\}$ ends by an infinite sequence of 0's, some point M_i must belong to the stable manifold of the fixed point $\Gamma \cap \Pi^{\rm in}$ (which is defined as a unique horizontal surface passing through this point and invariant with respect to T^{-1}), i.e. $M_i \in \{v = 0\}$.

Hence, the sequences $\{M_i\}$ correspond to the trajectories of the original maps T and $\tilde{T}$ if and only if the corresponding sequence $\{\sigma_i\}$ does not end with an infinite sequence of 0's. If the sequence $\{\sigma_i\}$ ends with an infinite sequence of 0's, we will cut the sequence $\{M_i\}$ at the last point to which $\sigma_i = 1$ corresponds. The new sequence $\{M_i\}$ will be a trajectory of the original maps T and $\tilde{T}$ which ends on the surface $(T_{glo}T_{loc}T_S)^{-1}(W^s_{\rm loc} \cap \tilde{\Pi}^{\rm in})$. All this is now in a complete correspondence with the statement of the theorem: recall that one iteration of the map T corresponds to one round of an orbit of the flow near the loop Γ and one iteration of the map T_S corresponds to one round near the superhomoclinic orbit S. End of the proof.

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