C. Global Attractors and Stability

3.1 Global Attractors and Limits

Organizers : Jack K. Hale, Geneviève Raugel

Key note lecture

Compact attractors

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The first part of the talk will discuss the necessity of the classical hypotheses for existence of compact global attractors. The second part will concentrate on diffusively balanced conservation laws.

Invited lectures

Attractors of Parabolic Problems and Perturbations of the Diffusion

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We are interested in studying the continuity properties of the asymptotic dynamics for second order parabolic problems when the diffusion coefficient varies in a singular fashion.

For $\epsilon \in (0, \epsilon_0)$, for certain $\epsilon_0 > 0$, the family of problems we consider are of the type

$$\begin{cases} u_t^\epsilon - \operatorname{div}(a_\epsilon(x)\nabla u^\epsilon) + u^\epsilon = f(u^\epsilon), & \text{on } \Omega \\ a_\epsilon \dfrac{\partial u^\epsilon}{\partial n} = g(u^\epsilon), & \text{on } \partial\Omega \end{cases}$$

where $f, g : \mathbb{R} \to \mathbb{R}$ are locally Lipschitz functions, $\Omega \subset \mathbb{R}^n$ is a bounded and smooth domain and a_ϵ, the diffusion coefficient, satisfies that there exist positive constants m_0, M_ϵ such that $m_0 \leq a_\epsilon(x) \leq M_\epsilon$.

Under several growth restrictions on f and g we will pose this problem in an L^q or $W^{1,q}$ setting and by imposing some dissipativity assumptions we will prove the existence of a global attractor $\mathcal{A}_\epsilon$. We will also obtain uniform bounds on $\mathcal{A}_\epsilon$ in the $H^1(\Omega)$ and $C^0(\bar{\Omega})$-norm.

We are interested in the following perturbation of the diffusion

$$a_\epsilon(x) \xrightarrow{\epsilon\to 0} \begin{cases} a_0(x) & \text{uniformly on } \Omega_1 \\ \infty & \text{uniformly on compact subsets of } \Omega_0 \end{cases}$$

where Ω_1 is a subdomain of Ω. This kind of perturbation may appear, for instance, in composite material, where the diffusivity behavior of the material may differ significantly from one part to another.

The limiting problem that we obtain is given by

$$\begin{cases} u_t - \operatorname{div}(a_0(x)\nabla u) + u = f(u) \qquad \text{on } \Omega_1 \\ a_0 \dfrac{\partial u}{\partial \vec{n}_0} = g(u) \qquad \text{on } \partial\Omega \\ u_{|\Omega_0} =: u_{\Omega_0}, \quad (\text{constant}) \qquad \text{on } \Omega_0, \\ \dot{u}_{\Omega_0} + \dfrac{1}{|\Omega_0|}\displaystyle\int_{\Gamma_0} a_0 \frac{\partial u}{\partial \vec{n}} + u_{\Omega_0} = f(u_{\Omega_0}) \end{cases}$$

and this problem has an attractor $\mathcal{A}_0$ which lies in a bounded subset of $C^0(\bar{\Omega})$.

We will show that the family of attractors $\{\mathcal{A}_\epsilon\}_{\epsilon\geq 0}$ is upper semicontinuous at $\epsilon = 0$ in the $H^1(\Omega)$ and $C^0(\bar{\Omega})$-norms, that is

$$\sup_{u_\epsilon \in \mathcal{A}_\epsilon} \operatorname{dist}(u_\epsilon, \mathcal{A}_0) \xrightarrow{\epsilon\to 0} 0.$$

where dist is the distance either in $H^1(\Omega)$ or in $C^0(\bar{\Omega})$. Notice that to obtain the upper semicontinuity in the uniform topology a delicate analysis of the behavior of the solutions near $\partial\Omega_0$ is needed.

On the time-periodic solutions to the Navier-Stokes equations of compressible flow

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We prove the existence of the time-periodic solutions with given positive mass to the Navier-Stokes system of compressible isentropic flow:

$$\partial_t \varrho + \operatorname{div}(\varrho u) = 0,$$

$$\partial_t(\varrho u^i) + \partial_{x_j}(\varrho u^i u^j) - \mu\Delta u^i - (\lambda+\mu)\partial_{x_i} \operatorname{div} u + a\partial_{x_i}\varrho^\gamma = \varrho f^i(t,x),\ i = 1,2,3.$$

with a given time-periodic driving force f. The unknown density $\varrho(t,x)$ and the velocity $u(t,x) = [u^1, u^2, u^3]$ are functions of the time t and the spatial variable $x \in \Omega$, where Ω is a bounded regular domain. The homogeneous Dirichlet boundary conditions are imposed on u.

References

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Attractors for dissipative problems on unbounded domains

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We consider parabolic problems of reaction–diffusion type on large or unbounded spatial domains $\Omega \subset \mathbb{R}^d$. Under suitable conditions on the nonlinearity, global existence and the existence of an absorbing ball can be established by using weighted energy estimates with a weight decaying at infinity, [1,2,6,7]. The method of weighted norms can even improve existing bounds for the absorbing ball in the case of bounded domains, [7]. Compactness properties of the semiflow are obtained in the weaker toplogy induced by the weighted norm. In this way it is possible to construct a global attractor $\mathcal{A}^\Omega$ which is closed and bounded in $\mathrm{L}^\infty(\Omega)$ and compact and attracting with respect to the weighted norm $\|\cdot\|_\rho = \sup\{\,\rho(x)|u(x)| : x \in \Omega\}$, [4,6,8].

Using examples we illuminate the essential differences between the dynamics on bounded and on unbounded domains. For instance, systems which are gradient flows when considered on bounded intervals will display nontrivial recurrent behavior when considered on the real line with initial data in $\mathrm{L}^\infty(\mathbb{R})$. Another instance is convergence to a whole continuum of steady states like in diffusive mixing, [4].

The attractor $\mathcal{A}^\Omega$ can be compared with $\mathcal{A}^{\mathbb{R}^d}$ by extending all functions on Ω by 0 outside of Ω. Upper semicontinuity for Ω approaching $\mathbb{R}^d$, that is

$$\mathrm{dist}_{\|\cdot\|_\rho}(\mathcal{A}^{\Omega_n}, \mathcal{A}^{\mathbb{R}^d}) \to 0 \quad \text{for } \Omega_n \supset \{|x| > r_n\} \text{ with } r_n \to \infty,$$

is obtained easily. Lower semicontinuity is false in general. However, considering a joint limit in the sense of [5], where the time and the domains go to infinity simultaneously, gives a limit attractor $\mathcal{A}^*$ which coincides with $\mathcal{A}^{\mathbb{R}^d}$.

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The global attractor of the viscous Cahn-Hilliard homotopy

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For the viscous Cahn-Hilliard homotopy it is possible to ascertain detailed information on the equilibria which allows the relative energies and Morse indices to be calculated, and a Morse decomposition based on energy can be constructed. Additionally, near the nonlocal Allen-Cahn end of the homotopy, a weak lap number principle is proven. This information permits a partial unraveling of many of the connections on the global attractor and demonstrates that the structure of the global attractor in the metastable region can be distinctly different from earlier predictions for the Cahn-Hilliard equation in zero mean mass (spinodal) case. Implications for a model for shape memory materials known as Ericksen's bar will be discussed if time permits.

References

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Contributed talks

Global Attractor in Overdamped Dynamics of tilted Frenkel-Kontorova Models

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We study a model for nonlinear conduction in charge density wave materials:

$$m\ddot{u}_n + \dot{u}_n = -h_1(u_n, u_{n+1}) - h_2(u_{n-1}, u_n) + F,$$

where subscripts on h denote partial derivatives, $m \geq 0$, $F \geq 0$, $h_{12} \leq -c < 0$, and $h(x+1, y+1) = h(x, y)$.

For the gradient dynamics case $m = 0$, we proved in [1] that there is a globally attracting time-periodic orbit (in the sense that there is a $T > 0$ such that $u_n(t+T) = u_n(t)+1$ for all t and n), in each of the following classes of configuration if F is sufficiently large that there is no equilibrium in that class:

1. finite chain with free ends;

2. spatially periodic configurations: $u_{n+q} = u_n + p$ for all n, for some integers p, q;

3. weakly rotationally-ordered configurations of irrational mean spacing.

The key tool is the monotonicity of the flow with respect to a partial order on configurations (u_n).

In the overdamped case $0 \leq m \leq (4\mu)^{-1}$, where $\mu = \sup h_{11} + h_{22}$, the flow is again monotone, this time with respect to a certain partial order on sequences $((u_n, \dot{u}_n))$. I shall discuss the extent to which the above results generalise to this case.

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On equivalence of exponential and asymptotic stability under changes of variables

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Lyapunov's notion of (global) asymptotic stability of an equilibrium is a key concept in the qualitative theory of ordinary differential equations and nonlinear control. In general, a far stronger property is that of *exponential* stability, which requires decay estimates of the type "$\|x(t)\| \leq ce^{-\lambda t}\|x(0)\|$." In this talk, we show that, for differential equations evolving in finite-dimensional Euclidean spaces $\mathbb{R}^n$ (at least in dimensions $\neq 4, 5$) the two notions are one and the same under coordinate changes. The same statement holds if perturbed differential equations are considered

Of course, one must define "coordinate change" with care, since under diffeomorphisms the character of the linearization at the equilibrium is invariant. However, if, in the spirit of both structural stability and the classical Hartman-Grobman Theorem, we relax the requirement that the change of variables be *smooth at the origin*, then all obstructions disappear. Thus, we ask that transformations be continuously differentiable, and they have inverses that are continuously differentiable except possibly at the origin, and globally continuous.

Closely related to our work is the fact that all asymptotically stable *linear* systems are equivalent (in the sense just discussed) to $\dot{x} = -x$; see e.g. [1]. The basic idea of the proof is based upon projections on the level sets of Lyapunov functions, which in the linear case of course be taken to be quadratic. It is natural to use these ideas also in the general nonlinear case, and Wilson's paper [2], remarked that level sets of Lyapunov functions are always homotopically equivalent to spheres. Indeed, it is possible to obtain, in great generality, a change of coordinates rendering the system in normal form $\dot{x} = -x$ (and hence exponentially stable), and several partial versions of this fact have appeared in the literature.

It is perhaps surprising that, at least for unperturbed systems, the full result seems not to have been observed before, as the proof is a fairly easy application of results from differential topology. Those results are nontrivial, and are related to the generalized Poincaré conjecture and cobordism theory; in fact, the reason that we only make an assertion for $\neq 4, 5$ is closely related to the fact that the original Poincaré conjecture is still open.

References

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Limit sets for reaction transport equations

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Our aim is to propose a hyperbolic generalisation of the well-known reaction diffusion equation and to study the structure of its ω–limit sets.

We consider moving and interacting particles, characterised by their position $x \in \Omega \subset \mathbb{R}^n$ in space and by their direction of movement $v \in V = S^{n-1}$. The evolution of the density $u(t,x,v)$ is governed by the following isotropic reaction transport equation

$$\left(\frac{\partial}{\partial t} + v \cdot \nabla + \mu L\right) u = \frac{1}{|V|} f(\bar{u}),$$

combined with the initial condition $u(0,x,v) = u_0(x,v)$ and appropriate boundary conditions on $\partial\Omega$. Here μ denotes the scattering parameter, $\bar{u}(t,x) = \int_V u(t,x,v)dv$ is the total density, $|V|$ the measure of V, and $Lu = u - \frac{1}{|V|}\bar{u}$ is the local deviation from the mean. The linear part on the left hand side models the spreading of particles due to a transport process, the nonlinearity f contains the reaction laws. The name isotropic refers to the usage of the total density $\bar{u}$ both in the scattering and in the reaction part. Equations of this type were proposed in [3,4] to model interacting bacteria when their motion consists of straight runs interrupted by re-orientations.

The linear existence theory of transport equations is well-established, cf. for instance [1] and the references therein.

On suitably defined Hilbert spaces we prove global in time existence of solutions by an argument based on a Lyapunov function approach. Here we assume a dissipativity condition on f, namely $\limsup_{|z|\to\infty} f(z)/z < 0$.

The main step in proving that orbits are relatively compact uses the far-reaching regularity estimates in [2] for velocity averages in transport equations. By classical results on the ω–limit sets of orbits, cf. [5], we can conclude that $\omega(u_0)$ is non-empty, compact, and connected. Furthermore, the Lyapunov function ensures that any ω–limit set is contained in the set of equilibrium solutions of the above reaction transport equation.

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Asymptotic Behavior of Some Nonlinear Dissipative Hyperbolic Equations on $\mathbb{R}^N$

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We consider the semilinear hyperbolic problem

$$u_{tt} + \delta u_t - \phi(x)\Delta u + \lambda\, f(u) = \eta(x),\ x \in \mathbb{R}^N,\ t > 0,$$

with the initial conditions $u(.,0) = u_0$ and $u_t(.,0) = u_1$ in the case where $\delta > 0$, $N \geq 3$ and $(\phi(x))^{-1} := g(x)$ lies in $L^{N/2}(\mathbb{R}^N)$. The energy space $\mathcal{X}_0 = \mathcal{D}^{1,2}(\mathbb{R}^N) \times L^2_g(\mathbb{R}^N)$ is introduced, to overcome the difficulties related with the noncompactness of operators which arise in unbounded domains. We derive various estimates to show local existence of solutions and existence of a global attractor in $\mathcal{X}_0$. We show that the global attractor has finite Hausdorff dimension and we prove the existence of an exponential attractor of finite fractal dimension that attracts all solutions at exponential rate. The compactness of the embedding $\mathcal{D}^{1,2}(\mathbb{R}^N) \subset L^2_g(\mathbb{R}^N)$ is widely applied.

Global attractors of reaction-diffusion equations without uniqueness

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We study the existence of global attractors for reaction diffusion equations in which the nonlinearity is a difference of two maximal monotone maps.

Even though for these equations it is possible to prove the existence of at least one solution corresponding to each initial condition, this solution can be non-unique. So, this equation does not generate a semigroup of operators. Taking the union of all possible solutions of a certain class we define a multivalued semiflow. By applying the theory of attractors for multivalued semiflows we obtain under certain conditions the existence of a global compact attractor.

This result is applied to several physical processes such as a model of combustion in porous media and some models of transmission of electrical impulses in nervous axons.

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Geometry of heteroclinic cascades in scalar semilinear parabolic PDE

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We study the long-time behaviour of one-dimensional reaction-diffusion equations

$$u_t = u_{xx} + f(u, u_x, x), \quad x \in [0,1],\ t \in \mathbf{R}^+$$

with Neumann boundary conditions. Particular properties of the corresponding semi-flow on the space of x-profiles, such as the gradient-structure or the linearization being of Sturm-Liouville-Type, allow for a detailed geometrical description of the global attractor.

The main tool for such investigations, beside the general geometric theory for semilinear parabolic equations, introduced by D.Henry [1], are the nodal properties of the stationary solutions. G.Fusco and C.Rocha recognized that one can describe these nodal properties in terms of a permutation of the equilibrium solutions [2]. This permutation which is determined already by the stationary equation, an ODE, can provide crucial information about the long-time behavior of the full PDE.

The attractor consists of the unstable manifolds of all equilibria. These are partitioned into sets of heteroclinic orbits, connecting to the same equilibrium. The main problem is to understand the structure of these sets and how they are assembled in the attractor. A first important step in this direction was a result of C.Rocha and B.Fiedler [3], using Conley-Index and giving a criterion for these sets to be empty or not. But as an inspection of several examples shows, their result is not sufficient to provide a satisfactory geometrical description of the attractor.

Mainly we present a result, giving more insight in the structure of the sets of connecting orbits by taking into account the decomposition of stable and unstable manifolds according to different exponential rates. We give a criterion in terms of the permutation, deciding exactly in which strong-stable and strong-unstable manifold the heteroclinic connections between given equilibria take place. This information seems also to be crucial for the understanding of the attractor as a whole. The main ingredient of the proof will be a detailed study of the permutation of equilibria. Finally, we suggest a new concept for the equivalence of attractors.

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Posters

Optimal Stabilization of Almost Periodic Systems

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Consider a system of differential equations of perturbed motion

$$\dot{x} = X(t, x; u) \tag{1}$$

where $x = (x_1, ..., x_n)$, $X = (X_1, ..., X_n)$, $u = (u_1, ..., u_r)$ -is a control function, and $X(t, x; u)$-is an almost periodic function in t.

Suppose that function $X(t, x; u)$ is defined, continuous and satisfying Lipschitz condition in x in the domain

$$t \in R, \|x\| < H \ (H = const). \tag{2}$$

Suppose that criterion of motion $x(t)$ is given as

$$I = \int_{t_0}^{\infty} \omega(t, x_1[t], ..., x_n[t]; u_1[t], ..., u_r[t]) \, dt, \tag{3}$$

where $\omega(t, x, u)$ is nonnegative function.

Theorem. If there exist almost periodic in t, positive definite,

continuously differentiable function $V^0(t, x)$ and functions $u_j^0(t, x)$, which are satisfying in the domain (2) the following conditions:

1) function $w(t, x) = \omega(t, x; u^0(t, x))$ is nonnegative, and $w(t, x)$ may equal zero only in the points of set which does not include any samitrajectory of the system (1) $x(x_0, t_0, t), (t_0 < t < +\infty)$ entirely (except the trivial solution);

2) an equality

$$B[V^0; t, x; u^0(t, x)] = 0 \tag{4}$$

holds, where

$$B[V; t, x; u] = \frac{\partial V}{\partial t} + \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} X_i(t, x; u) + \omega(t, x; u) = \frac{dV}{dt} + \omega(t, x; u); \tag{5}$$

3) an equality

$$B[V^0; t, x; u] \geq 0, \tag{6}$$

holds for each control functions u_j, then functions $u_j^0(t,x)$ solve the problem of optimal stabilization and an equality

$$\int_{t_0}^{\infty} \omega(t, x^0[t]; u^0[t])\, dt = \min \int_{t_0}^{\infty} \omega(t, x[t]; u[t])\, dt = V^0(t_0, x(t_0)) \tag{7}$$

holds.

3.2 Nonautonomous Attractors

Organizer : Mark Vishik

Key note lecture

Attractors of Nonautonomous Evolution Equations

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1. Nonautonomous evolution equations and the corresponding processes. Time-symbols of such equations and their hulls. Uniform attractors for a family of processes with translation compact symbols. Uniform attractor for nonautonomous 2D Navier-Stokes system, reaction-diffusion systems, dissipative nonlinear hyperbolic equations and other systems.

2. Upper and lower bounds for the fractal dimension of the attractors for the equations with quasiperiodic symbols. Upper bounds for Kolmogorov ε-entropy of the uniform attractor for the above equations with translation compact forcing terms and interaction functions.

3. Trajectory attractor for evolution equations without uniqueness of solution to the corresponding Cauchy problem or for equations for which the uniqueness is not proved yet. Construction of trajectory attractor for 3D-Navier-Stokes system, for dissipative hyperbolic equation with arbitrary polynomial interaction term and for other equations. Convergence of the trajectory attractors of finite dimensional Galerkin approximations to the trajectory attractor of corresponding system.

4. Trajectory attractor for an evolution equation that singularly depends on a parameter. Trajectory attractor for an evolution equations with rapidly oscillating coefficients and their limit – the trajectory attractor of the averaged equations. Examples.

The exposition follows the papers of V.Chepyzhov and M.Vishik.

Invited lectures

Kolmogorov Epsilon-entropy of Attractors of Non-autonomous Evolution Equations

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It is well known that global attractors of many autonomous dissipative equations and systems of mathematical physics have finite fractal dimension. Among such equations are: the 2D Navier-Stokes system, the reaction-diffusion systems, the Ginzburg-Landau equation, the nonlinear hyperbolic equation with damping and others.

In this report we consider uniform attractors of the corresponding non-autonomous equations and systems whose coefficients, external forces or interacting functions depend on time explicitly. All time-dependent terms of a particular equation are called the time symbol of this equation. The symbol is a quasi-periodic function, or an almost periodic function, or a general translation compact function with values in an appropriate Banach function space. It is proved that the uniform attractors of the mentioned above non-autonomous equations have finite fractal dimension if the time symbols are quasi-periodic. Moreover the upper estimate of the fractal dimension of the uniform attractor has two summands. The first summand coincides with the upper estimate of the global attractor of the corresponding autonomous equations, while the second summand is the number of rationally frequencies of the quasi-periodic symbol of the equation. At the same time, examples show that the fractal dimension of uniform attractors of non-autonomous equations can be infinite, when the symbol is an almost periodic function of time. This observation leads to the study of the Kolmogorov ε-entropy of uniform attractor since the ε-entropy describes the complexity of infinite dimensional compact sets in functional spaces.

We prove the following theorem. If the time symbol of the non-autonomous equation is a general translation compact function then the ε-entropy of the uniform attractor does not exceed the sum of two terms: the first term is upper estimate of the global attractor of the corresponding autonomous equation multiplied with $\log_2\left(\frac{1}{\varepsilon}\right)$ and the second term is the ε-entropy of the hull of the initial time symbol of the equation measured on the finite time interval having the length of the order $\log_2\left(\frac{1}{\varepsilon}\right)$. We note that in general the second term is the main term of the estimate as $\varepsilon \to 0$. (In the quasi-periodic case this term has the form $k \cdot \log_2\left(\frac{1}{\varepsilon}\right)$, where k is the number of rationally independent frequencies of the time symbol). In particular, the functional dimension of the uniform attractor does not exceed the functional dimension of the hull of the symbol of the non-autonomous equation under consideration. These results are applicable to various equations and systems arising in mathematical physics.

This research is the joint work with Professor Mark Vishik.

Long time behavior in phase-field models with memory

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We present two thermodynamically consistent phase-field models with memory. One is based on the linearized version of the Gurtin-Pipkin heat conduction law (cf. [1]), the other is characterized by a linear heat conduction law of Coleman-Gurtin type (cf. [2]). The formulation of Cauchy-Neumann initial and boundary value problems for these evolution systems is framed in a history space setting. Namely, the summed past history of the temperature is regarded itself as a variable along with the temperature and the phase-field. Well-posedness results are discussed as well as long term behavior of solutions. In particular, regarding the first model, suitable conditions are stated in order to ensure the existence of a uniform absorbing set. In the second case, taking advantage of a higher dissipation in the equation for the temperature, we can prove the existence of a uniform attractor.

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Long time behavior of models of Cahn-Hilliard equations in deformable media

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We are interested in the study of the long time behavior of models of Cahn-Hilliard equations in elastic deformable continua. More precisely, we consider models of generalized Cahn-Hilliard equations that take into account the effects of internal microforces introduced by M. Gurting, coupled with the stationary Navier equation of linear elasticity (we assume that the displacement gradient is small and that the deformation is infinitesimal).

We prove that the variational formulation associated with the problem can be uncoupled, defining a type of system that we call weakly coupled (nonautonomous) system.

We define and study the notions of (uniform) attractor and of (uniform) exponential attractor for a weakly coupled system and then obtain the existence of finite dimensional uniform attractors for the weakly coupled Cahn-Hilliard equations.

Existence, uniqueness, and asymptotic behaviour of the solutions of nonlinear hyperbolic equations

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Two hyperbolic equations of Mathematical Physics are considered, the first related to traveling waves in a strip, while the second refers to nonlinear models of the vibrating string and membrane, obtained without making any assumptions on the "smallness" of the motion and assuming a not necessarily linear stress-strain relationship. In the first case, it is proved that the corresponding semigroup possesses a trajectory attractor, constituted by the union of all bounded solutions of an associated set of equations. Regarding the vibrating string and membrane models, it is possible to prove the existence of a global solution and of an absorbing set.

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Attractors for nonautonomous dynamical systems

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We introduce a general nonautonomous dynamical system on a fiber bundle. Let $P \neq \emptyset$ be the base of this fiber bundle and $X(p)$ denote the fibers which are assumed to be Polish spaces. This dynamical system is given by a *cocycle*

$$\begin{aligned}&\phi(t,p,\cdot) : X(p) \to X(\theta_t p)\\ &\phi(t+\tau,p,\cdot) = \phi(t,\theta_\tau p,\phi(\tau,p,\cdot)), \quad \phi(0,p,\cdot) = \mathrm{id}_{X(p)}\end{aligned}$$

for $p \in P$, t, $\tau \in \mathbb{R}^+$ where θ is a flow on P:

$$\theta_{t+\tau} = \theta_t \circ \theta_\tau, \qquad \theta_0 = \mathrm{id}_P$$

for t, $\tau \in \mathbb{R}$. Note we do not assume that P carries any structure. In particular, we do not suppose that P is compact. An attractor of the nonautonomous dynamical system is a multifunction $p \mapsto A(p)$ with compact images such that we have a *generalized invariance*:

$$\phi(t,p,A(p)) = A(\theta_t p).$$

In addition, we have *pull back convergence*:

$$\lim_{t\to\infty} \mathrm{dist}(\phi(t,\theta_{-t}p,D(\theta_{-t}p)),A(p)) = 0$$

where D is a multifunction contained in the *basin of attraction*. If P carries a measurable structure then A is called a *random attractor*.

Conditions for the existence and uniqueness of such an attractor will be formulated. We will apply this general result to show the existence of attractors for the nonautonomous and the stochastic Sine Gordon equation and the 3D stochastic Navier Stokes equation (see [1]). In another application we will prove the existence of attractors for numerical schemes with nonequidistant steps (see [2]). In addition, we will study Lyapunov functions for nonautonomous dynamical systems introduced above (see [3]).

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Contributed talks

Dynamics of the slightly compressible Navier-Stokes equations

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The subject of this talk is to compare the long time behavior of the solutions of a slightly compressible Navier-Stokes equations and the ones of the incompressible Navier-Stokes equations (INSE), in a two dimensional bounded set Ω. We conclude by comparing the bounds of the fractal dimensions of the global attractors related to the two considered systems.

The considered slightly compressible model of Navier-Stokes equations reads,

$$\begin{array}{l} \partial_t u^\varepsilon - \nu \triangle u^\varepsilon - u^\varepsilon \wedge \text{rot } u^\varepsilon + \nabla \pi^\varepsilon = f \\ \varepsilon^2 \partial_t \pi^\varepsilon + \text{div } u^\varepsilon = 0 \\ u^\varepsilon(0) = u_0^\varepsilon \quad \pi^\varepsilon(0) = \pi_0^\varepsilon, \end{array} \tag{1}$$

where $u \wedge \text{rot } u = \text{rot } u \ (u_2, -u_1)^t = (\partial_1 u_2 - \partial_2 u_1)\ (u_2, -u_1)^t$.

We consider Dirichlet boundary conditions on $\partial\Omega$. This is an approximation of (INSE) which corresponds formally to $\varepsilon = 0$ without forced initial data on the pressure.

The dynamic of the problem (1) was firstly studied in [3] but the size and the fractal dimension of the global attractor was not estimated uniformly with respect to ε. Here, we introduce new results [1] [2]. We denote by P the orthogonal Leray projection on the subspace of $L^2(\Omega)^2$ of free divergence functions and $u_{\|}^\varepsilon = (I-P)u^\varepsilon$,

Theorem 1 *For smooth enough initial data $(u_0^\varepsilon, \varepsilon\pi_0^\varepsilon)$ and for $\varepsilon \leq \varepsilon_0$ (ε_0 fixed), there exist δ and K, two nonnegative constants independent of ε, such that*

$$|u_{\|}^\varepsilon(t)|_2^2 + |p^\varepsilon(t)|_2^2 \leq 4(|u_{0\|}^\varepsilon|_2^2 + |p^\varepsilon(0)|_2^2)e^{-\delta t} + \varepsilon^2 K,$$

with $\nabla p^\varepsilon = \varepsilon\nabla\pi^\varepsilon + (I-P)\,(Pu^\varepsilon \wedge \mathrm{rot}\ (Pu^\varepsilon) + \nu \triangle Pu^\varepsilon)$.
For other boundary conditions, an asymptotic expansion (w. r. to ε) can be given,

$$u^\varepsilon_{\|} = \varepsilon^2 u^1_{\|} + ..., \quad Pu^\varepsilon = u + \varepsilon^2 u^1_{\perp} + ..., \quad \pi^\varepsilon = \pi + \varepsilon^3 p^1 + ...$$

where (u,π) are solution of (INSE), $u^1_{\|}$, p^1 are explicitly given in function of (u,π).

Theorem 2 *For an adequate norm and for* $\varepsilon \le \varepsilon_1$, ε_1 *small enough, the fractal dimension of the attractor* $\mathcal{A}^\varepsilon$ *in the variable* $(u^\varepsilon, \varepsilon\pi^\varepsilon)$ *is bounded by* $G^2 = \frac{|f|_2^2}{\nu^4\lambda_1^2}$.
The fractal dimension of the global attractor related to (INSE) is bounded by G. This difference is only due to the fact that the system (1) is not fully dissipative.

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Hausdorff dimension estimates by use of adapted Carathéodory outer measures and applications to invariant sets with an equivariant tangent bundle splitting

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The contribution is concerned with upper bounds for the Hausdorff dimension of a flow invariant compact set $K \subset M$ which arises from an autonomous C^2-vector field f on M. Special Carathéodory outer measures are constructed which majorize the Hausdorff measures (see [1]). They are defined via covering elements which are tubular neighborhoods of arcs of smooth curves to approximate the fiber structure of the sets. New coordinates are introduced in order to separate the components of the vector field which act normally to the flow lines. The Hausdorff dimension bounds are formulated in terms of the eigenvalues of the symmetric part $\frac{1}{2}\pi(p)(\nabla f(p) + \nabla f(p)^*)\pi(p)$ of the operator which generates the associated system in normal variations with respect to the direction of the vector field (see [2]). Here $\pi(p)$ is the orthogonal projector onto the linear subspace orthogonal to the vector $f(p)$ and $\nabla f(p)$ is the covariant derivative of f in p. Assuming special properties of the stable and unstable manifolds of equilibrium points the results are derived for vector fields having a finite number of such equilibrium points in the considered invariant set.

Suppose that the compact and flow-invariant set $K \subset M$ without equilibrium points possesses an equivariant tangent bundle splitting $T_K M = E^0 \oplus E^1 \oplus E^2$ with respect to the flow. Then an entropy term of the flow can be indroduced into the dimension estimates. Considering the long-time behavior of the flow dimension bounds are derived including uniform Lyapunov exponents with respect to the splitting of the tangent bundle.

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Posters

On the Bounded Solutions of Autonomous Ordinary Differential Equations

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Consider an autonomous system of ordinary differential equations

$$\frac{dx}{dt} = X(x), \tag{1}$$

where $x, X \in R^n$, t is time, X is continuous function, satisfying a Lipschitz condition on any compact. Denote

$$x(t) = x(x_0, t) \tag{2}$$

the solution of the system (1) with initial data $x(0) = x_0$. Suppose that the semitrajectory (2) is bounded under $t \geq 0$, and all its ω-limit points are initial points of stable semitrajectories. Then $x(t) = y(t) + z(t)$, where $y(t)$ is almost periodic function, and $z(t)$ is vanishing function, i.e. $\lim\limits_{t \to \infty} z(t) = 0$.

On the Periodic Solutions of Ordinary Differential Equations

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Consider a system of ordinary differential equations

$$\frac{dx}{dt} = X(t, x), \tag{1}$$

where $x, X \in R^n$; t is time, X is continuous function, satisfying a Lipschitz condition on any compact. Besides we assume the function $X(t, x)$ to be periodic in t with the period ω_1:

$$X(t + \omega_1, x) \equiv X(t, x).$$

Theorem. Let $x(t)$ be any periodic solution of the system (1) with the period ω_2. If ω_1/ω_2 is an irrational number, then for every t_0 the function $X(t, x(t_0))$ is constant.

3.3 Order-Preserving Systems

Organizer : Hiroshi Matano

Key note lecture

Order Preserving Systems in the Presence of Symmetry

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Given an equation with certain symmetry such as symmetry with respect to rotation or translation, an important questions is whether or not the symmetry of the equation is inherited by its solutions.

In this lecture, which is based on a joint work with Toshiko Ogiwara (Proc. Royal Soc. Edinburgh 129 (1999), Discrete and Continuous Dynamical Systems 5 (1999)), we will first discuss the above question in a general framework of order-preserving dynamical systems under a group action and establish a theory concerning symmetry or monotonicity properties of stable equilibrium points.

In short, our result states that if G is the group of symmetry and if u is a stable equilibrium point (or a fixed point) of the dynamical system in an ordered metric space, then either $Gu = \{u\}$ or Gu is a totally ordered set homeomorphic to $\boldsymbol{R}$.

The results we present here are extensions of those of Mierczyński - Poláčik and Takač in the late 1980's, in which they required that the dynamical system be strongly order-preserving and the group of symmetry be compact and connected. We do not require the compactness of the group, which makes the theory applicable to the study of travelling waves. We have also relaxed the assumption that the dynamical system be strongly order-preserving, which allows our theory to deal with some degenerate equations and equations on unbounded domains. It should be noted that our idea is closely related to the so-called sliding method introduced by Berestycki and Nirenberg. In other words, it is, in some sense, an abstract formulation of sliding method.

As applications of our theory, we will show, among other things, the following:

(i) rotational symmetry of solutions of nonlinear elliptic equations;

(ii) instability of periodic solutions for equations of surface motion;

(iii) monotonicity of travelling (or "pseudo-travelling") waves.

We will also show, under additional assumptions, that stability implies asymptotic stability (or stability with asymptotic phase). Using this general result, we can show uniqueness and asymptotic stability of travelling or pseudo-travelling waves for a class of nonlinear diffusion equations and systems, including degenerate ones.

These results for the travelling waves put the earlier results by Volpert-Volpert-Volpert, Vega, Chen and other people in a more unified perspective.

Invited lectures

Order-preserving random dynamical systems

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We present the general concept of an order-preserving (or monotone) random dynamical system and we apply well-developed deterministic ideas to study the long-time behaviour of the trajectories and the properties of random attractors of these systems. As a main example we consider the random dynamical system generated by a coupled system of reaction–diffusion equations of the form

$$\partial_t u_j = \Delta u_j + f_j(u_1, \dots, u_m) + \sum_{k=1}^{m} h_{kj}(u_j) \cdot \sigma_k(t, \omega), \quad j = 1, \dots, m,$$

in a smooth bounded domain $D \subset \mathbb{R}^d$ with the Neumann boundary conditions. Here Δ is the Laplace operator, $f = (f_1 \dots, f_m) : \mathbb{R}^m \mapsto \mathbb{R}^m$ is a C^1-mapping satisfying the so-called cooperativity condition, $h_{kj} : \mathbb{R} \mapsto \mathbb{R}$ are smooth functions and $\{\sigma_k(t, \omega)\}$ is a family of (generalized) random stationary processes (white noise case is not excluded). We give the conditions that guarantee the existence of random equilibria and we study their stability properties under various additional assumptions concerning the functions $f_j(u_1, \dots, u_m)$ and $h_{kj}(u_j)$. We note that in contrast with the deterministic autonomous or periodic case random order-preserving systems can possess nontrivial completely ordered omega-limit sets. Our consideration relies crucially on the concept of sub- and super- equilibria which for the deterministic case is well-known, and for the random case was introduced in [1]. We also use random and stochastic comparison principles.

The results presented are based on ideas and methods which were developed in collaboration with L. Arnold [1,2] and with P. Vuillermot [3,4].

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Principal spectrum for nonautonomous linear parabolic PDEs

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We generalize the concept of principal eigenvalue to the case of a nonautonomous linear parabolic partial differential equation

$$u_t = \Delta u + a(t,x)u, \quad t \in \mathbb{R},\ x \in \Omega,$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with sufficiently smooth boundary and $a\colon \mathbb{R} \times \bar{\Omega} \to \mathbb{R}$ is a bounded continuous function uniformly Hölder in both variables. The equation is complemented with Dirichlet boundary conditions.

The idea of the generalization is the following. Consider a linear skew-product semiflow generated by the family of equations

$$u_t = \Delta u + b(t,x)u, \quad t \in \mathbb{R},\ x \in \Omega,$$

where b belongs to the closure in the supremum norm of the set of all time-translates of a. For that semiflow there is a canonically defined one-dimensional invariant subbundle S ([2]), which can be characterized as the set of all global solutions that are of the same sign for all $t \in \mathbb{R}$ ([1]).

The *principal spectrum* of the equation is defined as the dynamical (Sacker–Sell) spectrum (see [3]) of the skew-product semiflow restricted to S.

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Stable subharmonic solutions of reaction-diffusion equations

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We consider the Neumann problem for time-periodic reaction-diffusion equations. The asymptotic behavior of most bounded solutions of such equations is governed by stable periodic solutions. We address the question whether a stable periodic solution can be subharmonic, that is, whether its minimal period can be larger than the period of the equation.

Special attention will be given to nonlinearities that are spatially homogeneous, that is, they do not depend on the spatial variable explicitly. We show that, unlike in the nonhomogeneous case, the answer depends on the shape of the domain. Stable subharmonic solutions cannot occur on a convex domain, but they do occur on some nonconvex domains.

Posters

Star Product for Bimodal Maps

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This communication will discuss a symbolic star product of the dynamics of iterated cubic maps from the real line to itself. We also characterize the subsets in the parameters space that are self-similar.

3.4 Qualitative Theory of Parabolic Equations

Organizer : Peter Poláčik

Invited lectures

On the gradient structure of the FitzHugh–Nagumo system

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We consider the system of partial differential equations

$$\left\{\begin{array}{ll} u_t = d_1\Delta u + f(u) - v & \\ v_t = d_2\Delta v + \delta u - \gamma v & x \in \Omega. \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x) & \end{array}\right. \tag{1}$$

with Neumann or Dirichlet boundary conditions on $\Omega \in \mathbb{R}^n$ and sufficient conditions on f to ensure the existence of a global attractor for the semigroup generated by (1). A typical nonlinearity f is the cubic function $f(u) = \mu(u - u^3)$.

Due to the abundance of results on gradient-like systems, the existence of a Lyapunov functional for (1) has been considered in the literature for particular parameter regimes.

For example, if $d_2 = 0$ the system generated by (1) is gradient-like in the region $\delta \le \gamma^2$ and the same result holds for a shadow system obtained from (1) by a limiting process as $d_2 \to +\infty$.

We present a Lyapunov functional for (1) holding in the general case ($0 < d_1, d_2 < +\infty$) and draw some immediate consequences for the structure of the attractor.

This is a collaboration work with Pedro Freitas.

Reaction-diffusion equations on thin domains

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We consider reaction-diffusion equations on thin domains obtained by squeezing a given domain in some space direction by a factor $\epsilon > 0$. We show that as $\epsilon \to 0$ the corresponding dynamical systems tend to a limit system, generated by a parabolic equation on a graph. We also obtain the upper-semicontinuity of attractors as $\epsilon \to 0$, thus extending some earlier results of Hale and Raugel proved for domains lying under the graph of a function. (Joint work with Martino Prizzi)

Reaction-Diffusion Systems with Skew-Gradient Structure

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We consider the reaction-diffusion system

$$\begin{cases} Tu_t = D\Delta u + f(u) \quad \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{RD}$$

Here Ω is a bounded domain in $\mathbb{R}^N$, $u = u(x,t) \in \mathbb{R}^n$, $T = diag(\tau_1, \cdots, \tau_n)$ with $\tau_i > 0$, $D = diag(d_1, \cdots, d_n)$ with $d_i > 0$, and $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ is a smooth function of u. We say that the above system has *skew-gradient* structure if the nonlinear term f is written as

$$f(u) = Q\nabla_u H(u)$$

with some function $H(u) : \mathbb{R}^n \mapsto \mathbb{R}$ and $n \times n$ matrix $Q = diag(1, \cdots, 1, -1, \cdots, -1)$. In short, the skew-gradient system is a sort of activator-inhibitor system which resembles a gradient system but has nonlinearities with different sign.

In this talk, we discuss some fundamental properties of reaction-diffusion systems with skew-gradient structure. Any stationary solution of the skew-gradient system (RD) corresponds to a critical point of the functional

$$J[v] = \int_\Omega \left\{ \frac{1}{2}\langle D\nabla_x v, Q\nabla_x v\rangle - H(v) \right\} dx.$$

We study its stability as a steady state of (RD) and min-maximizing property as a critical point of $J[v]$.

Contributed talks

Logarithmic Sobolev inequalities and the rate of convergence to equilibrium for parabolic equations of Fokker-Planck type

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In this talk we are concerned with the large-time behavior of the Cauchy problem for linear Fokker–Planck type equations (advection-diffusion equations). These problems appear as homogeneous versions of kinetic Fokker–Planck models, e.g. In particular, we use the entropy method to analyze the rate of convergence to the equilibrium. Specifically, we consider the IVP for the *Fokker-Planck type equation*

$$\rho_t = \operatorname{div}(\nabla\rho + \rho\nabla A), \quad x \in \mathbb{R},\, t > 0; \quad \rho(t=0) = \rho_I \in L^1_+(\mathbb{R}^n)$$

with a given potential $A = A(x)$ such that the steady state $\rho_\infty = e^{-A} \in L^1(\mathbb{R})$.

For a wide class of problems the solution converges (in a weighted L^2-space) exponentially to the unique steady state, which is due to the spectral gap of the Hamiltonian that governs the evolution of the symmetrized problem. To extend this result to more general initial data we analyze the time decay of the *relative entropy* $e(t) = e_\psi(\rho(t)|\rho_\infty) = \int_{\mathbb{R}} \psi(\frac{\rho(t)}{\rho_\infty})\,\rho_\infty(dx)$, generated by the convex function $\psi(\sigma)$, $\sigma > 0$ with $\psi(1) = \psi'(1) = 0$. The idea of our analysis is to derive a differential inequality for the entropy production $I(t) = I_\psi(\rho(t)|\rho_\infty) = \frac{d}{dt} e_\psi(\rho(t)|\rho_\infty) \le 0$ and the entropy production rate $I'(t)$. For uniformly convex $A(x)$, i.e. $\left(\frac{\partial^2 A(x)}{\partial x^2}\right) \ge \lambda \mathbf{I}$, $x \in \mathbb{R}$, we get

$$\frac{d}{dt} I(t) = -2\lambda I(t) + r_\psi(\rho(t)), \tag{1}$$

with a non-negative remainder $r_\psi(\rho(t))$. This shows the exponential decay of $I(t)$. An integration in t then gives

$$I(t) = e'(t) \le -2\lambda e(t), \tag{2}$$

and the relative entropy converges to 0 exponentially with rate -2λ.

(2) is the entropy version of a convex Sobolev inequality. For the physical entropy ($\psi(\sigma) = \sigma\ln\sigma - \sigma + 1$) and Gaussian steady states $\rho_\infty = M_a$ (with standard deviation a) (2) can be transformed to the well-known *Gross logarithmic Sobolev inequality* (L. Gross, *Lecture Notes in Mathematics*, **1563**, 1993):

$$\int_{\mathbb{R}} f^2 \ln\left(\frac{f^2}{\|f\|^2_{L^2(dM_a)}}\right) M_a(dx) \le 2a \int_{\mathbb{R}} |\nabla f|^2 M_a(dx), \quad \forall f \in L^2(dM_a).$$

From the precise form of the remainder r_ψ in (1) one can deduce sharpness conditions on ρ, ρ_∞, and ψ such that (2) becomes an equality.

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The stability of solutions of the differential equations

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In [1] the stability criteria for solutions of systems of differential equations in the critical case of one zero root, based on analysis of spectrum of the Jacobi matrix for the right-hand side of the equation in a neighborhood of the solution is received. This method was generalized for investigation of stability of solutions of differential and difference equations in a Banach spaces and in all possible critical cases [2]. In the paper [3] this generalization was used for investigation of a domain of stability of systems of differential equations. In the paper [4] the Aizerman's problem was decided for a self-adjoint matrix. In the papers [5,6] the problems of stability of differential equations with lateness, of stability of the differential equations with a small parameter attached to derivative are investigated. In this paper we diffuse these results for receiving of some sufficient conditions of stability of differential equations of parabolic and hyperbolic types. In particular we study systems of partial differential equations of the following type

$$\begin{cases} \dfrac{\partial u_1(t,x_1,x_2)}{\partial t} = a_{11}(t)\dfrac{\partial^2 u_1(t,x_1,x_2)}{\partial x_1^2} + a_{12}(t)\dfrac{\partial^2 u_1(t,x_1,x_2)}{\partial x_2^2} + \\ \quad + a_{13}(t)\dfrac{\partial^2 u_2(t,x_1,x_2)}{\partial x_1^2} + a_{14}(t)\dfrac{\partial^2 u_2(t,x_1,x_2)}{\partial x_2^2} + g_1(t,x,u), \\ \dfrac{\partial u_2(t,x_1,x_2)}{\partial t} = a_{21}(t)\dfrac{\partial^2 u_1(t,x_1,x_2)}{\partial x_1^2} + a_{22}(t)\dfrac{\partial^2 u_1(t,x_1,x_2)}{\partial x_2^2} + \\ \quad + a_{23}(t)\dfrac{\partial^2 u_2(t,x_1,x_2)}{\partial x_1^2} + a_{24}(t)\dfrac{\partial^2 u_2(t,x_1,x_2)}{\partial x_2^2} + g_2(t,x,u) \end{cases} \tag{1}$$

with starting conditions

$$u(t_0;x_1,x_2) = u_0(x_1,x_2). \tag{2}$$

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Periodic Orbits and Attractors for Autonomous Reaction-Diffusion Systems

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Let $n \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. It is a well known fact that bounded solutions of an autonomous reaction-diffusion equation of the form $u_t = \lambda u_{xx} + q(u)$, $t > 0$, $x \in \Omega$ with Dirichlet-boundary conditions and $\lambda > 0$ tend to stationary solutions. Far less is known about solutions of systems of autonomous reaction-diffusion equations such as

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \lambda\Delta \begin{pmatrix} u \\ v \end{pmatrix} + g\begin{pmatrix} u \\ v \end{pmatrix}, \quad t > 0, \quad x \in \Omega \tag{1}$$

with Dirichlet boundary conditions and $\lambda > 0$. In order to understand the dynamics of reaction-diffusion systems, the description of the dynamics of the model system

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \lambda\Delta \begin{pmatrix} u \\ v \end{pmatrix} + f\begin{pmatrix} u \\ v \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} -v \\ u \end{pmatrix}, \quad t > 0, \quad x \in \Omega \tag{2}$$

can be looked at as a first step. We will see that the dynamics of (2) seriously differs from the dynamics of one equation; in particular, periodic motion can occur.
If $\dim \Omega = 1$, i.e. Ω is an interval, then we can prove a Poincaré-Bendixson result for (2) which means that all solutions tend either to zero or to a periodic orbit. We examine these periodic orbits and prove qualitative statements on the dynamics of all solutions of (2) using torsion number results. This is a concept for the examination of symmetric reaction-diffusion systems which is related to oscillation numbers for one equation.
If $\dim \Omega > 1$, then the existence of a stable periodic orbit can be proved under appropriate smoothness and monotonicity conditions on f. Furthermore, we can describe a part of the global attractor including the stable periodic orbit, the zero solution and all connections between them.

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Recents Results in Micromagnetism

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In this work, joint with Gilles Carbou, we study a model of ferromagnetic material governed by a nonlinear Laudau-Lifschitz equation coupled with Maxwell equations (1).

$$\begin{gathered} \tfrac{\partial u}{\partial t} + u \wedge \tfrac{\partial u}{\partial t} = 2u \wedge (\alpha\Delta u + H) \ \text{ in } \mathbb{R}^+ \times \Omega, \\ \mu\tfrac{\partial(H+\bar{u})}{\partial t} + \operatorname{curl} E = 0, \ \ \varepsilon\tfrac{\partial E}{\partial t} - \operatorname{curl} H = 0 \text{ in } \mathbb{R}^+ \times \mathbb{R}^3, \end{gathered} \tag{1}$$

with appropriated boundary conditions and initial data. Let us assume that

$$u_0 \in I\!H^1(\Omega)\ ,\ H_0 \in I\!L^2(\mathbb{R}^3)\ ,\ E_0 \in I\!L^2(\mathbb{R}^3),\ \ |u_0| = 1\ a.e.\ ,\ \operatorname{div}(H_0 + \bar{u}_0) = 0. \qquad (\mathcal{H})$$

This system describes electromagnetic waves propagation in a ferromagnetic medium confined to the domain Ω. In the ferromagnetic model the magnetic moment denoted by u links the magnetic field H with the magnetic induction B through the relationship $B = \mu(H + \bar{u})$. Moreover u is a vector field which takes its values on the unit sphere of $\mathbb{R}^3$. Finally ε is the dielectric permittivity and μ is the magnetic permeability. This model is described in detail in [2]. The quasistatic model corresponds to $\mu = 0$, $\varepsilon = 0$. After proving that under the assumption $(\mathcal{H})$, equation(1) has at least one weak solution, we establish :

Theorem 1 *Under assumption $(\mathcal{H})$, if u is a weak solution of (1), each point v of the ω-limit set of the trajectory u is a weak solution of the steady state system.*

Theorem 2 *Consider a sequence $(\varepsilon(n),\ \mu(n))_n$ which tends to $(0,0)$ as $n \to +\infty$ with $\mu(n)/\varepsilon(n)$ bounded. Then under the assumption $(\mathcal{H})$ if u^n denotes a weak solution of (1) with $\varepsilon = \varepsilon(n)$ and $\mu = \mu(n)$, there exists a subsequence $(u^{n_k})_k$ converging to u, solution of the quasistatic model.*

Theorem 3 *Let us assume $u_0 \in H^2(\Omega)$, $\left.\frac{\partial u_0}{\partial \nu}\right|_{\partial\Omega} = 0$, $|u_0| \equiv 1$. Then there exists a time $T^\star_\alpha > 0$ depending only on the size of the data, and a unique u such that for any $T < T^\star_\alpha$, $u \in \mathcal{C}^0\left((0,T); H^2(\Omega)\right) \cap L^2\left((0,T); H^3(\Omega)\right)$.*
Moreover $\lim\limits_{\alpha\to 0} T^\star_\alpha = +\infty$ and for any $T > 0$, $u_\alpha \to u$ in $L^2\left((0,T); H^2(\Omega)\right)$ with

$$\frac{\partial u}{\partial t} - u \wedge H(u) + u \wedge (u \wedge H(u)) = 0 \text{ in } \mathbb{R}_t \times \Omega \qquad (2)$$

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Connecting equilibria by blow-up solutions

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Radial solutions of the equation

$$u_t = \Delta u + \lambda e^u$$

on an N-dimensional ball are studied for $3 \leq N \leq 9$. We discuss blow-up-connections between equilibria. By a blow-up-connection from an equilibrium ϕ^- to an equilibrium ϕ^+ we mean a function $u(\cdot,t)$ which is a classical solution on the interval $(-\infty, T)$ for some $T \in \mathbb{R}$ and blows up at T but continues to exist in an appropriate weak sense for $t \in [T, \infty)$ and satisfies

$$u(\cdot,t) \to \phi^{\pm} \quad \text{as} \quad t \to \pm\infty$$

in a suitable sense.

On a Class of Selfsimilar Solutions of a Third-Order Model for the Shearless Turbulence Mixing Layer

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We study similarity solutions in the problem on the interaction and mixing between two semi-infinite homogeneous turbulent flow fields of different scales. The modelling study of this turbulence mixing layer is based on using a third–order model. This model includes a transport equation for the vertical component $< w^2 >$ of the turbulent kinetic energy and two equations of diffusion type for the triple correlation $< w^3 >$ and for the spectral flux ϵ of the turbulent kinetic energy. We analyze this model in the case of planar geometry. A self-similar solution of this system is a solution of the form

$$< w^2 >= t^{-2\mu} f(\xi), \quad < w^3 >= t^{-3\mu} g(\xi), \quad \epsilon = t^{-3\mu-\nu} h(\xi),$$

$$\xi = t^{\mu-1}(z - z_0), \quad z_0 = \delta_0 t^{1-\mu} + \delta_1, \quad \delta_k \equiv const, \quad k = 1, 2.$$

The free similarity exponent μ has to be determined from a solution of the obtained nonlinear eigenvalue problem. This is a typical situation appearing in problems of nonlinear diffusion where a conservation law does not exist. The boundary conditions are determined by the physical model, the functions $f(\xi)$, $g(\xi)$, $h(\xi)$ tend to given, generally speaking, positive limits at $\xi \to \pm\infty$. To find a solution to this nonlinear eigenvalue problem, we note that self-similar solutions can be found as steady-state solutions of the original system of partial differential equations after rewriting it in the new independent variables $x = t^{\mu}(z - z_0)$, $\theta = \log t$.

In the present article, we study the steady-states solutions and prove that there exists a positive steady state. The existence of a steady-state solution of this type implies that on this solution the system of equations will be transformed, returning to the original variables, into a system of two equations for $< w^2 >$ and ϵ in general. For the function

$< w^3 >$, we get a relation which is similar to a diffusion relation in the algebraic triple-correlation model. In other words, we prove that there is a regime of the turbulent flow which can be described in the framework of more simple model of second-order.

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Bifurcation of an equilibrium point in singularly perturbed parabolic system with transformed argument

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We consider the system of singularly perturbed parabolic equations with transformed argument

$$L(\varepsilon)\partial u/\partial t = D(t)\partial^2 u/\partial x^2 + A(t)u + B(t)u_\Delta + f(t,x,u,u_\Delta) \tag{1}$$

with periodic condition $u(t,x+2\pi) = u(t,x)$. Here ε is a small positive parameter, $L(\varepsilon) = diag[I_m, \varepsilon I_p], m+p=n$, $u_\Delta = u(t, x-\Delta)$, Δ is a transformation of the argument, matrices $D(t)$, $A(t)$, $B(t)$ and function $f : \mathbb{R}^{2n+2} \to \mathbb{R}^n$ are 2π periodic with respect to t, $f(t,x,u,v) = O(|u|^2+|v|^2)$ when $|u|+|v| \to 0$, the matrix $L^{-1}(\varepsilon)D(t)$ is positive definite.

We will search the solution of system (1) in the form of Fourier series

$$u(t,x) = \sum_{k=-\infty}^{\infty} y_k(t)\exp(-ikx). \tag{2}$$

Substituting (2) into (1), we obtain a countable system of differential equations in Fourier coefficients.

Let the following conditions are valid.

1) All roots of the characteristic equations $det(M_{k4}(t) - \lambda I) = 0$ lie on the halfplane $Re\lambda < 0$, where $M_k(t) = -k^2D(t)+A(t)+B(t)exp(ik\Delta)$, $M_{k1}(t)$, $M_{k2}(t)$, $M_{k3}(t)$, $M_{k4}(t)$ are blocks of matrix $M_k(t)$.

2) The monodromy operator for linear equations $dv_k/dt = P_k(t,\varepsilon)v_k(t)$, $P_k(t,\varepsilon) = M_{k1}(t)+M_{k2}(t)h_0(t)+\varepsilon M_{k2}(t)h_1(t)$, $h_0(t) = -M_{k4}^{-1}(t)M_{k3}(t)$, $h_1(t) = M_{k4}^{-1}(t)\ [dh_0/dt + h_0M_{k1}(t) + h_0M_{k2}(t)h_0]$, has a pair of roots $exp[\alpha(\varepsilon) \pm i\beta(\varepsilon)]$, $\alpha(0) = 0$, $\alpha'(0) \neq 0$, $\beta(0) \neq 0$, and the remaining roots satisfy $|\lambda| \neq 1$.

Since the conditions 1, 2 are fulfilled, the center manifold of the countable system of differential equations exist. If $\alpha'(0)$ and the first Liapunov value for the equation on manifold have opposite signs, an invariant torus of system (1) exist. It will be conditionally stable if $\alpha'(0) > 0$. The solutions on the torus may be periodic or quasiperiodic in dependence of the rotation number. These solutions are obtained by normal forms. If the rotation number on the torus satisfies some condition then subfurcation of periodic solutions occurs.

Inertial Manifolds and a Modified Strong Squeezing Property

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We study dissipative nonlinear evolution equations

$$\dot{u} + Au = f(u)$$

in a Banach space X. We are interested in the existence of positively invariant, exponentially attracting, smooth, finite dimensional manifolds M, the so-called *inertial manifolds*, see [1]. Usually inertial manifolds are seeked as the graph of a smooth function ϕ over PX where P is a finite projector.

It appears that independent of the method used a special condition is utilized predominantly of the form

$$\Lambda_2 - \Lambda_1 > CL(\Lambda_1^\nu + \Lambda_2^\nu),$$

usually referred to as a *spectral gap condition.* This spectral gap condition assumes that the spectrum of A is split into a finite set of eigenvalues with real part less or equal to Λ_1 and the remaining set of eigenvalues with real part greater or equal to Λ_2, and that the difference of Λ_2 and Λ_1 is large enough to majorize $CL(\Lambda_1^\nu + \Lambda_2^\nu)$, where L is a Lipschitz constant for f, $\nu \in [0, 1[$ is an additional parameter depending on f, and C is a technical constant depending on the particular proof used.

Since the spectral gap condition is the most restrictive condition with regards to applications, it is of interest to weaken this condition or even to find sharp conditions.

One way to weaken the assumptions is to work out the essential properties of the semiflow which are used for the construction of the inertial manifold. Such properties are the *cone invariance* and the *squeezing property.* The cone invariance property describes the fact that the difference of solutions cannot leave a certain cone. This property is mainly used to prove the existence of the manifold. Robinson [2] has shown that the cone invariance property is sufficient to prove the existence of an invariant Lipschitz manifold. The squeezing property requires that the difference of each pair of solutions decays exponentially as long as the difference is outside of a possibly another cone; it is used to prove the attraction properties. Together both properties are called *strong squeezing property*, see [2].

We shall replace the strong squeezing property by introducing a *modified strong squeezing property* consisting of a cone invariance property and a modification of the squeezing property. To assume this property is a natural assumption and it is also sufficient for the existence of an inertial manifold having the *asymptotic completeness property.* We show that in general the modified strong squeezing property is weaker than the strong squeezing property.

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Banach contraction principle in parabolic problems

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The differential-functional problem

$$D_t u(t,x) \; - \; \sum_{j,l=1}^{n} a_{jl}(t,x)\, D_{x_j x_l} u(t,x) \; = \; f\left(t, x, u_{(t,x)}\right)$$

is transformed to a fixed point equation $u = \mathbb{T}\, u$ which is solved by means of the iterative method $u^{(k+1)} = \mathbb{T}\, u^{(k)}$. The Banach contraction principle (=BCP) provides the error estimates

$$\|u^{(k+2)} \; - \; u^{(k+1)}\| \; \leq \; \theta \, \|u^{(k+1)} \; - \; u^{(k)}\|, \qquad \theta \in (0,1),$$

provided the Lipschitz condition is satisfied. We discuss the question how far the BCP is from nonlinear cases.

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Unfocused blow up in semilinear parabolic equations

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The aim of this talk is to study the blow up behaviour of radially symmetric solutions u of a semilinear parabolic equation, around a blow up point other than its centre of symmetry. We show that u behaves as if a one-dimensional problem was concerned, that is the possible blow up patterns around an unfocused blow up point are the ones corresponding to the case of dimension $N = 1$.

Collision of Layers in a Scalar Reaction-Diffusion Equation of 1-Space Dimension

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We are concerned with the following scalar reaction-diffusion equation:

$$u_t = \epsilon^2 u_{xx} + u(u-a)(1-u), \quad -\infty < x < \infty,$$

where $\epsilon > 0$ and $0 < a < 1/2$ are parameters. One can observe that for small ϵ, transition layers emerge from appropriate initial data and those move with speed $O(\epsilon)$ as long as each layer is apart from the others of distance $O(1)$. This dynamics of the layers was well studied mathematically. It is also known by numerical computations that two facing layers collide and eventually collapse in a finite time. In this lecture we show rigorously the collision and the collapse of any two facing layers starting from a suitable initial function. We also estimate the life time of the layers. A key tool used in the study is a lower solution globally defined for $t > 0$. This is joint work with M. Mimoto [2].

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Formation of stable inner layers in diffusion equations: necessary and sufficient conditions

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In [1], S.B. Angenent, J. Mallet-Paret and L.A. Peletier classified all stable stationary solutions of a one-dimensional space semilinear boundary value problem which appears as a simplified model in population genetics. Their problem naturally generalises to N-space dimension as follows:

$$\frac{\partial v_\varepsilon}{\partial t} = \varepsilon^2 \triangle v_\varepsilon + v(1-v)[v-a(x)]$$

with $\partial v_\varepsilon / \partial \hat{n} = 0$ on $\partial\Omega$ and $\Omega \subset \mathbb{R}^N$. In particular, in [1] every solution develops inner transition layer with interface located at the zeros of $a(x) - (1/2)$. For the ease of presentation we work with the nonlinearity $f(x,v) = (v+1)(1-v)[v-a(x)]$.

We know from [2] that given a smooth hypersurface S in Ω, one of the necessary conditions for a family v_ε of stationary solutions of the above equation to develop an inner transition layer, as $\varepsilon \to 0$, with interface S is that the N-vector equation holds

$$\int_S a(x)\hat{n}(x)dS = 0$$

where $\hat{n}(x)$ stand for the outward normal vector on S.

Given a nodal hypersurface S of $a(x)$ we will give sufficient conditions on a for the existence of a family of solutions as above having S as interface. We suppose that S lies between two arbitrarily narrow N-dimensional annulus on each of which $a(x)$ is radially symmetric and satisfies two technical conditions. Our method of proof relies on techniques based on Γ-convergence and on sub- and super-solutions. More specifically we take two ε-families of conveniently defined functionals whose Γ-limits, as $\varepsilon \to 0$, have each of them, an isolated local minimiser in $BV(\Omega; \{\alpha(r), \beta(r)\})$, the space of functions of bounded variation in Ω which assume only two values: $\alpha(r)$ and $\beta(r)$. Here $\alpha(r) = (1/3)[a(r) - (3a^2(r)+9)]^{1/2}$ and $\beta(r) = (1/3)[a(r) + (3a^2(r)+9)]^{1/2}$. Close to each of these minimisers (in the L^1-topology) there is a local minimiser of the original functional. These, by its turn, will generate a sub- and a super-solution of the original equation and existence of a solution, for ε small, follows.

Sufficient conditions for the existence of such solutions for equations like the above one but with variable diffusivity and the nonlinearity $f(x,v)$ satisfying the generalized equal-area condition $\int_a^b f(x,\xi)d\xi = 0, \ \forall x \in \Omega$, where a and b are zeros of f (not necessarily consecutive), will be provided.

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On the stabilization of bounded solutions to parabolic systems with analytic nonlinearity and Liapunov functionals

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In this communication we consider a boundary value problem to parabolic systems in bounded domains. We assume that the problem has analytic dependence of unknown function and its derivatives. We also assume that the problem has a Liapunov functional. Let $u(x,t)$ denote the solution of problem in the classical sense. We say that the solution $u(x,t)$ stabilizes (stabilizes in backward time) if $u(x,t)$ converges as $t \to +\infty (t \to -\infty)$ to one stationary solution. It is shown that the uniformly bounded for $t \geq 0 (\leq 0)$ solution to the problem stabilizes (stabilizes in backward time). Let us assume that the problem is dissipative. It is shown that the global attrator of this problem consists of stationary solutions and connected orbits and the flow on global attrator is gradient like. In contrast, for one parabolic equation with many space variable (and of course for parabolic system with one space variable) with Liapunov functional the stabilization theorem becomes false. The example of such nonconvergent bounded solution was constructed by P.Polćhik and K.Rybakovskii. This example shows that it is impossible to omit the assumption about the analytic dependence of nonlinearity.

3.5 Stability of Fronts and Pulses

Organizer : Björn Sandstede

Key note lecture

Dynamics of pulses and fronts

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Travelling waves form an important class of patterns that arise in a variety of applied problems. The issue considered in this talk is the interaction between several well-separated stable travelling waves. Suppose that a certain dissipative partial differential equation posed on either the real line or an unbounded cylinder exhibits a stable pulse. We begin by describing the ordinary differential equations that govern the interaction of finitely many, well-separated copies of that pulse [1]. Equilibria of these differential equations correspond to multi-hump pulses whose stability properties were studied, for instance, in [5,6,7,8]. If infinitely many pulses are concatenated, the situation is far more complicated. As a first step towards a more complete description, the stability of spatially periodic wave trains with large period is considered. It has been proved by Gardner that their spectrum consists entirely of essential spectrum; here, we locate their spectrum [2]. A different type of interaction occurs for travelling pulses near the onset to instability. Suppose that the homogeneous asymptotic rest state towards which the pulse converges destabilizes. This bifurcation is often referred to as the Turing bifurcation; typically, small spatially-periodic stationary patterns are created near the asymptotic rest state. We show that the interaction between the pulse and these small Turing patterns produces a continuum of time-periodic travelling pulses that resemble a superposition of the pulse and the Turing patterns [3,4]. The main technique used in the proofs of the aforementioned results are exponential dichotomies together with Lyapunov-Schmidt and center-manifold reductions.

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Invited lectures

The stability of large pulse solutions

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First, it is shown that there exist various families of large-amplitude homoclinic (multi-)pulse solutions in singular perturbed reaction-diffusion equations (on the real line) under relatively mild conditions. Second, the stability of these pulses is investigated using the Evans function. It is established that the natural slow/fast decomposition of the Evans function exhibits singularities: the slow component can have poles at (some of) the zeros of the fast component. The eigenvalues of the linearized stability problem, i.e. the zeros of the Evans function, can be determined explicitly by the recently developed 'NLEP method'; this method gives a leading order approximation of the slow component of the Evans function. As a consequence, it can be shown that only the 1-pulse solutions can be stable: all homoclinic multi-pulse solutions must be unstable. The methods will be applied to the Gierer-Meinhardt problem.

Convergence to travelling waves in damped hyperbolic equations

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We discuss the stability of travelling waves for a nonlinear damped hyperbolic equation on the real line. This system depends on a characteristic time $\eta > 0$ as a parameter, and reduces to the parabolic Fisher equation as $\eta \to 0$. Using energy estimates and a hyperbolic version of the maximum principle, we show that the travelling waves are stable against perturbations in a weighted Sobolev space. In the important case of fronts travelling with minimal speed, we prove in addition that the perturbations decay to zero like $t^{-3/2}$ as $t \to +\infty$ and approach a universal self-similar profile, which is essentially independent of η, of the nonlinearity, and of the initial data.

Slowly-modulated two-pulse solutions and pulse-splitting in the 1-D Gray-Scott model

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Two pulse solutions play a central role in the phenomena of self-replicating pulses in 1-D reaction-diffusion systems. In this talk, which is on a joint paper with Arjen Doelman and Wiktor Eckhaus, we focus on the 1-D Gray-Scott model as a prototype. We report on an existence and stability study for solutions consisting of two pulses moving apart from each other with slowly varying velocities. In the various parameter regimes, critical maximum wave speeds are identified, and ODE's are derived for the wave speed and for the separation distance between the pulses. The bifurcations in which these solutions are created and annihilated, and in which they gain stability, are determined. Good agreement is found between these theoretical predictions and the results from numerical simulations. The inhibitor component is far away from its homogeneous value, and slowly varying, between the pulses. Hence, the results presented here apply to the strong pulse interaction problem. The main methods used are analytical and geometric singular perturbation theory for the existence demonstration, and the nonlocal eigenvalue problem (NLEP) method developed in our earlier work on stationary pulses for the stability analysis.

Essential instabilities of travelling waves

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We analyze instabilities of travelling waves in reaction-diffusion systems caused by the essential spectrum crossing the imaginary axis. As a typical example, we examine travelling fronts and pulses whose asymptotic states undergo a Turing instability. Under typical assumptions we prove bifurcation of stable modulated travelling waves asymptotic to small amplitude Turing patterns, if the Turing instability is created ahead of the front. If the instability is created behind the front, we show that modulated travelling waves cannot bifurcate and the primary front remains convectively stable. A pulse is shown to move through a Turing pattern, leaving a recovery zone of Turing patterns, which are growing in amplitude, behind.

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Existence and stability of modulating fronts

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Modulating front solutions look like a front like envelope advancing in the laboratory frame and modulating an underlying spatially periodic pattern. They are transient solutions which connect a spatially periodic equilibrium with the unstable spatially periodic ground state. A classical example is the bifurcating front connecting the Taylor vortices with the trivial Couette flow for the Taylor–Couette problem in infinite cylinders. Here we talk about existence and stability of such objects.

Contributed talks

Multi-pulse solutions in problems with the reflection symmetry in an unbounded domain

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Let us consider an evolutionary autonomous system that is invariant under shifts $z \mapsto z + const$ in the unbounded spatial coordinate z and reflection $z \mapsto -z$. We assume that the basic fully symmetrical steady state loses stability via perturbations with the spatial critical wave number $k_c = 0$. That is the case for a variety of the physical problems, for instance: 3D Poiseuille problem on viscous incompressible fluid flows between parallel plates, convection in inclined layer, film flows, chemical reactions and so on (see [1],[2]). If the time periodicity is presumed together with periodicity in additional unbounded directions then near the instability threshold the local study of the problem can be reduced with the use of the spatial dynamics formulation to the analysis of the phase portrait of $SO(2)$-invariant reversible vector field with nonsemisimple eigenvalue 0 of the linearized problem. Quasihomogeneous truncation corresponding to a positive face of the Newton polyhedron of the ODE problem is equivalent to the steady complex Ginzburg-Landau (cGL) equation ([2,3,5]. For the cGL equation a primary family of homoclinic orbits $T_\alpha H_0$, $\alpha \in S^1$ is known explicitly. Under a generic transversality condition with respect to parameter, the existence of homoclinic n-pulse solutions is demonstrated for a sequence of parameter values. The existence of cascades of $2^l 3^m$-pulse solutions follows by showing their transversality and then using induction. The method relies on the construction of an $SO(2)$-equivariant Poincaré map which, after factorization, is a composition of two involutions: a logarithmic twist map and a smooth global map. Reversible periodic orbits of this map corresponds to small reversible periodic or homoclinic solutions of the problem. Using perturbation arguments we relate reversible n–pulse solutions of this equation to n–pulse solutions of the original problem on a spatial center manifold. Thus, we obtain multi–pulse solutions of the PDE problem for parameters near the criticality. In 3D Poiseuille problem these solutions are localized in spanwise direction, but periodic in streamwise direction.

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Stability of travelling fronts for reaction-diffusion-convection systems

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We are concerned with the role of travelling fronts in the asymptotic behaviour of solutions of systems of the form

$$u_t = Au_{xx} + f(u, u_x),\ x \in \mathbb{R}, t > 0, u(x,t) \in \mathbb{R}^N,$$

$$u(x,0) = u_0(x),\ x \in \mathbb{R},$$

where A is a positive-definite diagonal matrix and the nonlinearity f has two ordered equilibria and satisfies conditions sufficient for a comparison principle for this system to hold.

Suppose that f is "bistable" and that there exists a travelling-front solution w with velocity c connecting the two stable equilibria of f. (We have a set of hypotheses on f under which such a front is known to exist.) If u_0 is bounded, uniformly continuously differentiable and such that $\|w(x) - u_0(x)\|$ is small when $|x|$ is large, we can show that there exists $\xi \in \mathbb{R}$ such that

$$\|u(\cdot,t) - w(\cdot - ct + \xi)\|_{BUC^1} \to 0 \quad \text{as} \quad t \to \infty,$$

(see [1]). Our approach extends an idea developed by Roquejoffre, Terman and Volpert in the convectionless case, where f is independent of u_x. First local stability is proved using the spectrum of the linearisation about w. Then u_0 is assumed to be increasing in x, and convergence proved via a homotopy argument. The result for arbitrary u_0 is deduced by showing that there is an increasing function in the ω-limit set of u_0.

Suppose now that f is "monostable". In this case, there is some $c_0 \in \mathbb{R}$ such that there exist travelling-front solutions for each velocity $c \geq c_0$. We are investigating the stability of the supercritical waves ($c > c_0$) by formulating the problem in exponentially weighted function spaces, where the velocity-dependent weight is chosen so as to render the "monostable" nonlinearity essentially "bistable", and thus amenable to some of the methods used in the "bistable" case. (Such use of weighted spaces goes back to work of Sattinger, and has been discussed by Volpert, Volpert and Volpert for reaction-diffusion systems where f is independent of u_x.)

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Global structure of traveling waves for 3 competing species model with diffusion

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Reaction-diffusion equations are used extensively as continuous space-time models for interacting and diffusing biological species in population dynamics. In mathematical ecology, there are three types of interactions between individuals with the diffusion terms which model the migration of each species; competition for limited resources, prey-predator interaction, and mutualistic relationships. Though these equations are relatively simple, they can exhibit a variety of interesting spatial and spatio-temporal patterns, including traveling fronts and pulses.

In this talk, we study the following 3-component reaction-diffusion systems for three competing species :

$$(1)\quad \left\{\begin{array}{lclcl} u_{1t} & = & d_1 u_{1xx} & + & (r_1 - a_{11}u_1 - a_{12}u_2 - a_{13}u_3)u_1 \\ u_{2t} & = & d_2 u_{2xx} & + & (r_2 - a_{21}u_1 - a_{22}u_2 - a_{23}u_3)u_2 \\ u_{3t} & = & d_3 u_{3xx} & + & (r_3 - a_{31}u_1 - a_{32}u_2 - a_{33}u_3)u_3 \end{array}\right. \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{R},$$

where $u_i(t,x)$ denote the population densities of three competing species at time t and spatial position x. d_i are the diffusion rates, r_i are the intrinsic growth rates, and a_{ij} are the intraspecific and interspecific competition rates, respectively. All of the coefficients are positive constants. We assume that

(H1) $d_1,\ d_2\ <<\ d_3$ (i.e., $d_1 = \varepsilon^2, d_2 = \varepsilon^2 d, d_3 = 1$ for small positive ε),
(H2) If $u_1 = 0$, this system (1) has a stable equilibrium solution $P_2 = (0, u_{*2}, u_{*3})$ with $u_{*2} > 0$ and $u_{*3} > 0$. On the other hand, if $u_2 = 0$, (1) has a stable equilibrium solution $P_3 = (u_1^*, 0, u_3^*)$ with $u_1^* > 0$ and $u_3^* > 0$.

Let us consider the following situation. If the u_1-species invades on the stable equilibrium state P_2, what happens with the population dynamics of the invader u_1-species ? For suitable parameters, we can check numerically that u_1-species propagates into the state P_2 with constant shape and constant velocity and inhibits u_2-species and then it reaches to the state P_3. This motivates us to study the existence and stability of traveling fronts connecting the states P_2 with P_3. We use the analytical singular perturbation method to show the existence and the SLEP method to show the stability. Finally, we consider the global structure of traveling wave solutions; traveling fronts, standing pulses, traveling pulses, with respect to some parameter.

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Instability of Multiple Pulses in a System of Quadratically Coupled Schrödinger Equations

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Many mathematical models studied in modern optics take the form of two or more Schrödinger equations coupled together in a nonlinear fashion. Of special interest are solitary-wave solutions for which all components of the system are localised in time (representing pulses in optical fibres) or in the space directions transverse to propagation (modelling beams in waveguides). This talk is mainly concerned with a system that describes the phenomena of second-harmonic generation and parametric wave interaction in dispersive quadratic media. In particular, the existence and stability properties of multiple pulses (multibump solitary-waves) are investigated within the framework of homoclinic bifurcation theory, employing a functional-analytic approach with strong geometric underpinnings. All of the multiple pulses generated via the bifurcation occurring at a resonant semi-simple eigenvalue scenario are demonstrated to be unstable; we then discuss the generalisation and implications of this result for other, related, systems.

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