D. Computational Aspects

4.1 Computer Algebra Tools

Organizer : Jan Sanders

Key note lecture

Algorithms from representation theory

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First a description of the role of $sl(2, \mathbb{R})$ in the computation and classification of nilpotent normal forms is given. This leads to a similar role for the Heisenberg algebra in the case of the total differentiation operator

$$\mathcal{D}_x = \frac{\partial}{\partial x} + \sum_{i=0}^{\infty} u_{i+1} \frac{\partial}{\partial u_i}.$$

One define derivations

$$\mathcal{F}_x = \sum_{i=1}^{\infty} i u_{i-1} \frac{\partial}{\partial u_i}.$$

and

$$\mathcal{E}_x = \sum_{i=0}^{\infty} u_i \frac{\partial}{\partial u_i}.$$

These form a Heisenberg algebra with the usual operator commutation rules. Using representation theory one can now effectively *integrate*, i.e. solve f from $g = \mathcal{D}_x f$. One can now define transvectants or Hirota operators by

$$\tau_n(f, g) = \sum_{r+s=n} \frac{(-1)^r}{r!s!} \mathcal{D}_x^r \mathcal{E}_x^s(f) \mathcal{D}_x^s \mathcal{E}_x^r(g) = \frac{1}{n!}(\mathcal{E}_x \wedge \mathcal{D}_x)^n f \otimes g.$$

Notice that $\tau_n : \ker \mathcal{F} \otimes \ker \mathcal{F} \to \ker \mathcal{F}$, i.e. the bilinear operators are defined on the space of functionals and τ_0 and τ_1 define a Poisson structure there.

One can then deform the usual multiplication by defining

$$f \star_t g = \sum_{n=0}^{\infty} t^n \tau_n(f, g).$$

This defines an associative product, and thereby a noncommutative deformation of the Poisson structure.

The various connections between classical invariant theory, transvectants, Hirota operators, Rankin-Cohen brackets in the theory of modular functions and pseudo-differential operators and the Poisson algebra of functionals, as they are defined in the theory of integrable evolution equations will be discussed.

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Invited lectures

Algorithmic invariant theory and dynamics

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In recent years there has been a lot of progress in computational algebra, especially in the algorithmic treatment of invariant rings and modules of equivariants. The aim of this talk is to show the benefits of this for situations in the theory of equivariant dynamical systems where a general equivariant vector field is required. I will discuss the consequent algorithmic treatment of the center manifold reduction with symmetry. The method of Noether normalization will be explained and the advantages of using Noether normalization in this context illustrated.

Finally, we will comment on a generalization of the orbit space reduction to group actions on the differential equations which may act on the independent variable and may be nonlinear.

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Using the symbolic method to classify evolution equations

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The main purpose of this talk is to demonstrate the existence of (infinitely many) symmetries and conservation laws for λ—homogeneous evolution equations using the symbolic method.

The symbolic method first introduced by Gel'fand and Dikii. The idea is simply to replace u_i, where i is an index — in our case counting the number of derivatives — by ξ^i, where ξ is now a symbol. The basic operation of differentiation, i.e. replacing u_i by u_{i+1}, is now replaced by multiplication with ξ, as is the case in Fourier transform theory. For higher degree terms with multiple u's, one uses different symbols to denote differentiation.

The key fact is the following symbolic formula for the bracket of a differential polynomial Q of degree $\mathbf{m}$ with a linear differential polynomial:

$$\widehat{[u_k, Q]} = G_k^{(m)} \, \widehat{Q},$$

where

$$G_k^{(m)} = \xi_1^k + \cdots + \xi_m^k - (\xi_1 + \cdots + \xi_m)^k.$$

With this one can readily translate solvability questions into divisibility questions solved by diophantine approximation theory, and we can use generating functions to handle infinitely many orders at once.

The complete list of λ—homogeneous integrable evolution equations contains only 10 all-known equations when $\lambda > 0$. This explains why after the gold rush it was so difficult to find any new integrable systems.

The classification of co-symmetries (conservation laws) is more complicated. One often encounters equations with only a finite number of conservation laws. Here a complete classification for 5-th order KdV-like equations is given using generating function.

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A Method for Computing Center Manifolds and Normal Forms

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Center manifold theory and normal form theory are two basic tools in the study of nonlinear differential equations. The idea of these two theories is to use a sequence of near identity nonlinear transformations to obtain a simple form on the so called center manifold. This reduction greatly simplifies the analysis of the dynamical behavior of the system such as bifurcations and instability. Different methodologies have been developed for computing normal forms, based on matrix theory, algebraic manipulation, representation theory, invariant theory, etc. [1–4]. However, for a given system, finding the explicit formulas of normal forms and the associated nonlinear transformations is not easy. Thus, symbolic computer languages such as Maple, Mathematica and Macsyma have been introduced in such computations.

Normal forms are generally not uniquely defined due to the existence of uncertainty in computing the coefficients of nonlinear transformations. This difficulty may be overcome either by imposing additional conditions so that the coefficients of nonlinear transformations can be uniquely determined; or by imposing certain conditions such that some coefficients in the normal forms may be set zero. Such a simplification raises a question: In general how to find a "simplest" or "minimum" normal form? Some results related to simple singularities have been obtained ([5, 6], more references can be found in [5]). However, it is noted that a *unique* normal form may be not a "simplest" form, and simplest normal forms may be not unique.

We will present an early developed perturbation technique for computing the normal forms of a general n-dimensional system described by a set of differential equations for certain cases of singularities [7]. The method does not need the application of center manifold theory, nor requires for solving large matrix systems. In fact, the approach combines the normal theory and center manifold theory in one computation procedure. Moreover, it has been found that the method may be extended to directly compute a unique, "simplest" normal form for certain cases. Symbolic computation using Maple will be discussed, and examples will be presented to show the computation efficiency of the method.

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Contributed talks

Computer Algebra Tools for the Solution of Boundary-value Problems

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The introduced package provides tools for numerical solution of linear boundary value problems, both regular and eigenvalue ones. It has a syntax of the built-in *Mathematica* function **NDSolve** and strongly enhances its functionality. For the solution of mentioned problems two basic methods are applied. The first one is known as *Chasing* method and was generalized by authors particularly for multipoint problems, and the second one is the well-known *Shooting* method. The newest version of *Mathematica* 3.0 contains the function **NDSolve** which has a very limited ability to cope with regular boundary value problems, but not with eigenvalue ones. Unfortunately, the function uses an in some sense erroneous algorithm which leads sometimes to strange and even wrong results. Moreover, it can solve neither boundary value problem containing more than one equation nor multipoint ones. Below we present an example of a wrong result: **NDSolve** can't cope with the following problem: the command

$$\mathbf{NDSolve}[\mathtt{y}'''[\mathtt{x}] + \mathtt{y}[\mathtt{x}] == \mathtt{0}, \mathtt{y}'[\mathtt{0}] == \mathtt{0}, \mathtt{y}'[2] == \mathtt{1}, \mathtt{y}''[\mathtt{0}] == \mathtt{0}, \mathtt{y}[\mathtt{x}], \{\mathtt{x}, \mathtt{0}, \mathtt{2}\}]$$

gnereates the answer: `NDSolve::"unsol": "Not possible to initiate boundary value problem with the chasing method"` - message from **NDSolve**, while **DSolve** gives analytical solution:

$$\mathbf{DSolve}[\mathtt{y}'''[\mathtt{x}] + \mathtt{y}[\mathtt{x}] == \mathtt{0}, \mathtt{y}'[\mathtt{0}] == \mathtt{0}, \mathtt{y}'[2] == \mathtt{1}, \mathtt{y}''[\mathtt{0}] == \mathtt{0}, \mathtt{y}[\mathtt{x}], \mathtt{x}]//\mathtt{Simplify}$$

$$\{\{y(x) \to \frac{\left(-i+\sqrt{3}\right)\, e^{2+i\sqrt{3}-\frac{\left(3+2\,(-1)^{\frac{2}{3}}\right)x}{2}}\, \left(e^{\frac{3x}{2}} + e^{\frac{i}{2}\sqrt{3}\,x} + e^{\frac{(3+2\,i\sqrt{3})\,x}{2}}\right)}{-2\,i\,e^{3} + i\,e^{i\sqrt{3}} - \sqrt{3}\,e^{i\sqrt{3}} + i\,e^{3+2\,i\sqrt{3}} + \sqrt{3}\,e^{3+2\,i\sqrt{3}}}\}\}$$

In contrast, our implementation of the abovementioned methods produces more accurate results for a wider class of problems. In addition to the solution of a regular boundary value problem our functions can solve eigenvalue problems, which is very important in applications (for instance, for linear stability analysis). We present an example of linear stability analysis in Marangoni convection in the spherical layer, which was successfully performed using our tools. Another advantage of the presented tools is also the possibility to input the problem in its natural form.

A computational method for the stability of a class of mechanical systems

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(Joint work with Prof. **Ângelo Barone-Netto** and Prof. **Mauro de Oliveira Cesar**, of the Institute of Mathematics and Statistics, University of São Paulo, Brazil).

Consider the differential system

$$\begin{aligned} \ddot{x} &= -xf(x) \\ \ddot{y} &= -yg(x), \end{aligned}$$

where $f, g\colon \mathbb{R} \to \mathbb{R}$ are smooth functions and $f(0) > 0$, $g(0) > 0$. Clearly the system has the constant solution $x(t) \equiv y(t) \equiv 0$. The problem here is how to decide the stability of this equilibrium solution, that is, what is the asymptotic behaviour as $t \to +\infty$ of the trajectories starting with initial data close to 0.

We will say that the system associated to a given pair of functions f, g is k*-decidable* for some $k \in \mathbb{N}$ if the stability of the equilibrium can be decided from the values of the derivatives up to the order k of f and g at 0. The system will be finitely decidable if it is k-decidable for some k.

Not all pairs of functions are finitely decidable. However finite decidability is generic in C^∞, and we describe an algorithm for finding explicit sufficient conditions for k-decidability. When implemented on a computer the algorithm runs easily enough for $k = 1$ up to 5 or 6.

With the explicit conditions at our disposal it is easy to construct examples of pairs f, g and of sequences of initial data converging to 0 for which either all $y_n(t)$ are unbounded as $t \to +\infty$, or, more interestingly, where $t \mapsto y_n(t)$ is bounded for all n, but $\sup_{t\geq 0} |y_n(t)|$ diverges as $n \to +\infty$.

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4.2 Control and Optimization

Organizer : Fritz Colonius

Key note lecture

Dynamical Aspects of Control

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We consider families of ordinary differential equations where time dependent coefficients are allowed to take their values in a given set. Thus one obtains a family of nonautonomous differential equations indexed by the coefficient functions which may be considered as (open loop) admissible controls. The long time behavior of this family can be described via an associated skew product flow, the control flow, over the base space of admissible controls. The topologically transitive components and the chain transitive components correspond to the control sets and the chain control sets, respectively. If the closure of a control set is a chain control set, then it depends continuously (in the Hausdorff metric) on parameters. Also some related bifurcation problems will be discussed.

Invited lectures

Characterization and Computation of Reachable Sets via Optimal Control

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For controlled and perturbed systems governed by nonlinear dynamics of the form

$$\dot{x}(t) = f(x(t), u(t))$$

on some manifold M, with $u(\cdot)$ from some function space $\mathcal{U}$, the knowledge of reachability properties plays an important role in the analysis of the system behaviour.

For $u(\cdot)$ being a control function e.g. the *reachable set*

$$R(x) := \{y \in M \mid \exists u(\cdot) \in \mathcal{U} : \ \phi(t, x, u(\cdot)) = y\}$$

is an important object; if $u(\cdot)$ models a time varying perturbation, and $x^* \in M$ is a locally stable equilibrium for each $u(\cdot) \in \mathcal{U}$ the *robust domain of attraction*

$$RD(x^*) := \{y \in M \mid \phi(t, y, u(\cdot)) \to x^* \text{as}\, t \to \infty\, \forall u(\cdot) \in \mathcal{U}\}$$

is of interest [4].

In this talk we give a survey on recent work [2, 3] about the characterization of these sets via optimal control techniques. Appropriate formulations are e.g. minimum/maximum time and minimum/maximum distance optimal control problems, which have in common that their optimal value functions are characterized by Hamilton-Jacobi-Bellman PDEs. Furthermore we will present a generalization of the classical Zubov's equation [1] which nicely fits into the Hamilton-Jacobi framework.

Based on these formulations we then discuss numerical methods for the computation of these sets and illustrate them by a number of examples. In particular, we will highlight advantages and limitations of this approach compared e.g. with trajectoriwise or cell mapping algorithms.

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Twist maps and sliding-mode control

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In this talk we will study the dynamical stability properties of a suitably defined approximation of equivalent control in sliding manifold control systems. For this we use both the classical theory of singularly perturbed dynamical systems and the theory of twist maps. Examples of control problems which possess dynamical stability properties are presented. The work discussed here continues the investigation of control design, via singular perturbation theory, for control systems posessing a sliding manifold.

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Controllability of Matrix Eigenvalue Algorithms

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Numerical matrix eigenvalue methods such as the QR algorithm or inverse power iterations provide interesting examples of nonlinear discrete dynamical systems defined on Lie groups or homogeneous spaces. A typical approach from numerical linear algebra to improve convergence properties of such algorithms is via suitable shift strategies for the eigenvalues, see [1]. Such eigenvalue shifts can be viewed as control variables and the resulting algorithms can therefore be analyzed using tools from nonlinear control theory. So far the analysis and design of shift strategies in numerical eigenvalue algorithms has been more a kind of an art rather than being guided by systematic design principles. The advance made during the past two decades in nonlinear control theory indicates that the time may now be ripe for a more systematic investigation of control theoretic aspects of numerical linear algebra.

In this talk we investigate the controllability properties of the well known inverse power iteration for finding the dominant eigenvector of a matrix A, defined by

$$x_{k+1} = \frac{A^{-1}x_k}{\|A^{-1}x_k\|}.$$

Shifted versions of the inverse power method defined by

$$x_{k+1} = \frac{(A - u_k I)^{-1}x_k}{\|(A - u_k I)^{-1}x_k\|}$$

lead to nonlinear control systems on projective space, or more generally on Grassmann manifolds. For cyclic matrices A necessary and sufficient conditions for complete controllability on the projective space are given. These are always satisfied if complex shifts are allowed.

In the real case the situation is more involved and we use recent results on the existence of universally regular controls in our analysis, see [2]. In the real case necessary conditions in terms of the location of the eigenvalues are presented. These may be equivalently rephrased by the algebraic statement that every polynomial with real coefficients splits modulo the characteristic polynomial of the matrix A into linear factors over $\mathbb{R}$. We conjecture that these conditions are also sufficient. If this turns out to be true it may be inferred that complete controllability of the inverse power iteration is a generic property.

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Contributed talks

Controllability of generic control systems near k-singular points

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Control system on n-dimensional smooth manifold is defined by smooth family of vector fields parameterized by the points of $(n-1)$-dimensional sphere. A *generic* control system is a system from a certain open everywhere dense subset in the space of such families endowed with the fine Whitney's C^∞–topology.

At a point of the phase space values of the fields forms *velocity indicatrix* at this point, and any velocity from the indicatrix is admissible one at this point. Attainability of one point from another one is defined in standard manner in the class of piecewise continuous controls.

A point of the phase space has *small time local transitivity property* (=SLTP) if for any time $T > 0$ and any neighborhood V of this point there exists a neighborhood of this point such that any two point of the latter neighborhood are attainable from each other in a positive time smaller than T and along a trajectory lying in the neighborhood V. A *k-singular point* is defined as a point of the boundary of set of points having SLTP such that there is a passage through zero on a plane of support to the velocity indicatrix at this point containing exactly k admissible velocities. In the generic case there can appear such singular points with $k \in \{1, 2, \ldots, n\}$.

Theorem *For $n \geq 3$ any k-singular point of a generic control system on n-dimensional manifold has SLTP if $3 \leq k \leq n$ and it has nor SLTP if $k = 1$.*

For 2-singular points for a generic control system some necessary and some sufficient conditions to have SLTP are found. In [1], [2] the complete answer on local controllability of systems on surfaces was obtained for the generic case. In particular, their controllability near 1-, 2-singular points was studied (in [3] analogous results are obtained for a generic two dimensional dynamic inequality).

This research was supported in part by RFBR970100713, FCT and Calouste Gulbenkian Foundation.

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The descent constant in the optimal control problems with Lipschitzian differential inclusions

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We consider the optimal control problem with fixed time endpoints: $J(x(S), x(T)) \to \min$, $(x(S), x(T)) \in C$, $\dot{x}(t) \in F(x(t))$, $G(x(t)) \leq 0$, $\forall t \in [S,T]$ where minimum is sought in the class of absolutely continuous arcs $x(\cdot)$, $x(t) \in R^{d_x}$. Here J, G are real-valued Lipchitz functions; $F(\cdot)$ is Lipchitz multifunction with convex compact values belonging to R^{d_x}, C is close subset of $R^{d_x} \times R^{d_x}$.

For this problem the necessary condition of the strong minimum in the maximum principle (MP) form is well-known [1]. In the important subclass with locally-convex J, G where C and $F(x)$ can by represented by a regular system of equations and inequalities, this MP is equivalent [2] to existence of a nontrivial solution of Eulier equation, i.e. there is no reducing directions simultaneously tangentially to local problem's constrains. But in the case of general problem under consideration the connection of conjugate multipliers and geometrical "descent algorithm" is lost completely. This serious defect of modern MP theory is connected with ineffective traditional way of its derivation (penalty method, Ekeland theorem ets.). The purpose of proposed investigation consist in the restoring of the of noted relation. Consider, for brevity, the problem without state constraint $G \leq 0$ and define the auxiliary objects.

$\Lambda(x) := \{\lambda = (\alpha, l, n_C, \psi(t), n(t))\}$ - the totality of conjugate strings of an admissible trajectory $x(\cdot)$. Here $\alpha \geq 0$, $l \in \partial J(x(S), x(T))$, $n_C \in N_C(x(S), x(T)), n(t) = (n_x(t), n_u(t)) \in clgN_{gr\,F(x(t))}(x(t))$, $\psi(t)$, $n(t)$ are Lipchitz and Borelian measurable functions with values in R^{d_x}, $R^{d_x} \times R^{d_x}$ respectively; N_C, $clgN_{grF(x(t))}$ denotes the cones of external normals and the closure "be graph" of the external normals for the respect sets.

$b(x) := \inf\{\|\psi - n_x\|_{L_1} + \|\dot{\psi} - n_u\|_{L_1} + \|(\psi(S), \psi(T)) - n_c\| : \lambda \in \Lambda(x), \alpha = 0, \|(\psi, n, n_C)\| = 1\}$; $a(x) := \inf\{\|\psi - n_x\|_{L_1} + \|\dot{\psi} - n_u\|_{L_1} + \|(\psi(S), \psi(T)) - n_c - \alpha l\| : \lambda \in \Lambda(x), \alpha = 1\}$

Theorem : Let the admissible trajectory $x_0(\cdot)$ satisfy the inequalities $a(x_0) > 0, b(x_0) > 0$. Then $\forall \epsilon > 0\ \exists \delta > 0$ and a family of admissible trajectories $\{x_r\}$, $r \in [0, \delta]$ such that $\forall\, 0 \leq r' \leq r'' \leq \delta$ the following conditions are satisfied: $\|x_{r'} - x_{r''}\| \leq r'' - r'$, $J(x_{r''}(S), x_{r''}(T)) - J(x_{r'}(S), x_{r'}(T)) \leq -(a(x_0) - \epsilon)(r'' - r')$.

That is, the objective function can be decreased over the "Lipchitz" family of admissible trajectories x_r with the speed $a(x_0)$. The proof of theorem is based on constructive Lusternike's type iteration process. Note, that there is no, in general, the strict descent directions.

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About the Optimality Conditions in Control Problems

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Kossy Smatov

The Optimal Control Problems prescribed by Operator Relations in the Banach Spaces are considered in the Work. The Development of the absolute Minimum Theory based on the V.F.Krotov Lemma is given. Let $K = \{k|k = 1, 2, ...\}, S = \{s|s = 1, 2, ...\}$ are index Sets. We assume that the Families of Sets $\{D_k, k \in K\}, v_k \in D_k$ are given. The Functional $J_k : D_k \to R$ on D_k is given.

Lemma 1. Let the Problems $\{<< D_k, J_k >>, k \in K\}$ satisfy to the Conditions: 1^0. for all $k \in K : J_k(v) \geq J_{k+1}(v), v \in D_k$; 2^0. there are the Integers $p, q \in K$ and the Sequence $\{v_s, s \in S\} \subset D_p$ such that $p < q$ and $\lim_{s\to\infty} = i_q = \inf_{D_q} J_q$. Then $\{v_s, s \in S\}$ is minimizing for the Problems $\{<< D_k, J_k >>, k = p, p+1, ..., q\}$ and any minimizing Sequence for $<< D_k, J_k >>$ satisfies to the Condition 2^0 and is also minimizing for Family of Problems $\{<< D_k, J_k >>, k = p, p+1, ..., q\}$.

Let Y_1, Y_2, U_1 are the Banach Spaces, Y_1 is reflexive, continuously and densely embedding in $Y_2 . Y = Y_1^n, U = U_r$ with Elements $y = \{y_1, ..., y_n\}$ and $u = \{u_1, ..., u_r\}$ accordingly, $\{y, u\}$ are the pair State-Control. Let

$$Y_c \subset Y, \quad U_c(y) \subset U, \quad V_c = \{y, u|y \in Y_c, u \in U_c(y)\} \subset V = Y \times U. \tag{1}$$

We suppose that the pair State-Control satisfies to the following functional Constraints:

$$A_i(y, u) = 0_{Y_2}, i = 1, ..., l_1, \quad A_i(y, u) \geq 0_{Y_2}, i = l_1 + 1, ..., l. \qquad (2).$$

We designate a Set of pairs (y, u) which satisfy (1), (2) as D_1 and assume $D_1 \neq \emptyset$. Functional $J_1(y, u)$ is defined in D_1.

Problem. To find a Solution of the Minimization Problem $<< D_1, J_1 >>$.

Let

$$D_2 = \{y, u|(1)\}, \quad J_2(y, u) = J_1(y, u) - \sum_{i=1}^{l} < \lambda_i(y), A_i(y, u) >, \tag{3}$$

where $\lambda_i : Y \to Y_2, i = 1, ..., l_1, \quad \lambda_i : Y \to P^*, i = l_1 + 1, ..., l$, are nonlinear Operators, P is the Cone of positive Elements in Y_2.

Theorem 1. Let there are a). Operators $\lambda_i, i = 1, ..., l$; b). Sequence $\{y_s, u_s, s \in S\} \subset D_1$ such that

$$\lim_{s\to\infty} J_1(y_s, u_s) = i_2 = \inf_{D_2} J_2. \tag{4}$$

Then $\{y_s, u_s, s \in S\} \subset D_1$ is minimizing for $<< D_1, J_1 >>$, and any minimizing Sequence for the Problem $<< D_1, J_1 >>$ satisfies to the Condition (4).

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Bifurcation of control sets

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We consider control-affine systems on $\mathbb{R}^d$ of the following form:

$$\begin{array}{ll} \dot{x} & = f_0(x) + \sum_{i=1}^m u_i(t)\, f_i(x), \\ u & \in \mathcal{U}^\rho = \{u : \mathbb{R} \to \mathbb{R}^m, u(t) \in U^{\rho} \forall t \in \mathbb{R}, \text{meas.}\} \end{array} \quad (1)$$

where $U \subset \mathbb{R}^m$, compact and convex containing 0 and $U^\rho = \rho \cdot U$ for $\rho \in [0, \rho^*] \subset \mathbb{R}$. We assume, that the system (1) has a *singular point* $x^* \in \mathbb{R}^d$, i.e. $f_i(x^*) = 0$ for all $i = 0, \ldots, m$.

Control sets are subsets of $\mathbb{R}^d$ where every point can be steered to another at least approximately. Motivated by numerical experiments with the perturbed *Duffing-van der Pol* equation, the question arised under which conditions there are control sets around the singular point x^*. By linearization of the system (1) at the singular point we get the bilinear control system

$$\dot{x} = A_0 x + \sum_{i=1}^m u_i(t) A_i x, \quad u \in \mathcal{U}^\rho \text{ where } A_i := \left.\frac{\partial f_i}{\partial x}\right|_{x=x^*}. \quad (2)$$

Denote the solution by $\phi(t, p, u)$ with $\phi(0, p, u) = p$. For every $(u, p) \in \mathcal{U}^\rho \times \mathbb{R}^d$ define the *Lyapunov exponent* $\lambda(u, p) := \limsup_{t\to\infty} \frac{1}{t} \ln \|\phi(t, p, u)\|$. There is a Whitneysum-decomposition $\mathcal{U} \times \mathbb{R}^d = \mathcal{V}_1^\rho \oplus \ldots \oplus V_l^\rho, 1 \le l \le d$ such that $\Sigma_{Ly}^\rho(\mathcal{V}_i) = \{\lambda(u, p) : (u, p) \in \mathcal{V}_i^\rho\}$ are compact intervals (cf. [2]). Under regularity conditions on (1) and (2) and with

$$0 \in \text{int}\Sigma_{Ly}^\rho(\mathcal{V}_1) \text{ and } \Sigma_{Ly}^\rho(\mathcal{V}_i) \subset \mathbb{R}^- \text{ for } 2 \le i \le l,$$

we can show, that there exists a control set D with nonvoid interior and $x^* \in \text{cl}D$. Here we use the Hartman-Grobmann Theorem and the theory of stable and unstable integral manifolds for nonautonomous differential equations (cf.[1]) for constructing an appropriate control function. The control set D is characterized via the stable and unstable manifolds of (1). We show that there is a control set of the system (2) which is tangential to D at x^*.

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Generalized Solutions for Singular Linear-Quadratic Optimal Control Problems

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We study general linear-quadratic optimal control problems with time and end points fixed. It is well known that singular problems of this type may fail to have a minimizer in the class of "ordinary", for example, square integrable controls, even in cases when the problem has a finite infimum. In such cases, the minimizing sequences of trajectories for the problem may converge to a discontinuous curve, implying that the respective optimal controls contain impulses.

From the functional-theoretic point of view, the problem is one of finding the minimizer of a quadratic functional on an affine subspace, $\mathcal{U}$, of finite codimension in $L_2^k[0,T]$. The approach we suggest is the extension of the functional by continuity onto a larger class of controls and finding its minimizers in the extension (closure) of $\mathcal{U}$.

We find a set of conditions necessary and sufficient for the existence of an affine subspace of finite codimension in $L_2^k[0,T]$ on which the functional has finite infimum. These conditions are algebraic with respect to the initial data of the problem. We show that when these conditions are satisfied, there is a unique extension of the functional onto a certain subspace, H_{γ_r}, of the Sobolev space $H_{-n}^k[0,T]$, (n=dimension of the state space). If the functional has finite infimum in $\mathcal{U}$, then the extended functional has a minimum in the closure of $\mathcal{U}$ relative to the topology of H_{γ_r}. The problem of finding a minimizer of the extended functional is equivalent to a regular linear-quadratic optimal control problem, with final state confined to an affine subspace of $\mathbb{R}^n$ instead of a single point.

Under suitable conditions (finiteness of the infimum and controllability of the system), there exists a minimizer of the extended functional that is the sum of a real-analytic function on $[0,T]$ with a distribution of order $r \leq n$ concentrated at the initial and final instants. The trajectories corresponding to such controls are sums of real analytic functions on $[0,T]$ with distributions of order $(r-1)$. If the optimal control is unique, any minimizing sequence of "ordinary" trajectories must approximate the generalized optimal trajectory in the topology of $H_{-(r-1)}^n[0,T]$.

Assuming controllability of the system and existence of a solution for an appropriate Riccati differential equation, we provide an algorithm for the computation of generalized optimal controls.

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Exact boundary controllability for a forth order parabolic equation

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We consider exact boundary controllability problem for a forth order parabolic equation. We obtain exact boundary controllability results when the initial data is infinitely differentiable.

On dual approach to optimization problems with elliptic operator

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We study the optimization problem formulated on solutions of some family of boundary value problems. This family is generated by different boundaries (i.e. by smooth closed lines on the plane so that each of these lines bounds some coherent corresponding area. All boundary value problems have the differential equation generated by means of the same elliptic operator (in our example, it is Laplace's operator). The object function of this problem describes the uniformity of the vector potential with respect to the vertical component under restriction on the horizontal component values. Thus, we vary the boundary of the area in order to obtain the most uniform field, i.e. vector potential (in sense of the some measure) in the fixed subarea of the considered area.

The problem formulation describes an important practical problem, namely, creating the most uniform magnet field in MRI-tomography devices. To obtain results of the medical MRI-test which are coincident with the real state of patients, it is necessary to generate uniform magnet fields of high rate with respect to the vertical intensity. In addition, it is necessary to restrict the other component into some prescribed limits. It is well known that constant magnet fields have a scalar potential satisfying the Laplace equation.

We propose an effective numerical recursive algorithm which is based on a two-step procedure in each recursion. This procedure contains the boundary value problem with a modified boundary on each recursion (as the first step) and the special auxiliary linear programming problem (as the second step). The last one is intended for two aims: 1) to find the most convenient parts of the boundary for a variation into the current recursion and 2) to find the points of the subarea which influence most essentially an improvement of the object function. A dual form of the auxiliary problem has been a new approach to solving this multidisciplinary problems. The convergence theorem of the method and several results on the degree of convergence are obtained in this work. This approach may also be used in other boundary value problems with elliptic operators.

Flight Autopilot Design for a Remotely Piloted Vehicle using LQ and feedback controller theories

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In the last years, several control theories have been widely developed. They are generally applied to control task such as trajectory tracking and mission optimization. In this paper, we consider a Remotely Piloted Vehicle (RPV airplane), the linear quadratic (LQ) and feedback methods are used to perform the control law of the autopilot.

The equations of the complete airplane motion are first linearized and the flight parameters decoupled. The total motion is equivalent to a sum of superposed modes.

The LQ design of the autopilot consists in the minimization of a cost function. The result is then obtained by the solution of the Riccati equation. When using the feedback theory, the adjustement of the autopilot gains is based on the pole placement procedure. The response of the complete system (nonlinear RPV airplane with autopilot) is finally simulated. The two methods has proved to be well adapted for this design. The results show that the system is stable with adequate values of rapidity and damping ratio for all the flight modes.

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Stabilization of Overhead Traveling Cranes by Nonlinear Feedback

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Overhead traveling cranes are widely used in factories for the transfer of heavy loads. In our previous papers [1], [2], a set of nonlinear differential equations was derived as an exact mathematical model of the crane, which makes three-dimensional motion. In this paper, we present a solution to the problem of global feedback stabilization of the nonlinear system of the overhead traveling cranes. The crane system has three motors, and torques of the motors can be taken as the control inputs to the system. By using Lyapunov's direct method, we derive a nonlinear feedback control law that ensures the global asymptotic stability of the equilibrium state of the crane system. It will also be proved that the stabilizing control is of switching type.

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Local theory for differential equations of Carathéodory-type

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A nonautonomous differential equation $\dot{x} = f(t,x)$ is of *Carathéodory-type* if $f : D \subset \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is *measurable in* t (for fixed x) and C^k *in* x (for fixed t). Two examples: *Random differential equations* (RDE) $\dot{x} = f(\theta_t\omega, x)$ driven by a metric dynamical system θ are pathwise of Carathéodory-type [1]. *Bilinear control systems* $\dot{x} = [A_0 + \sum_{i=1}^N u_i(t)A_i]x$ for fixed controls $u : \mathbb{R} \to U$, $U \subset \mathbb{R}^N$ compact and convex, are of Carathéodory-type [4]. In this talk we present three new results:

1. For linear systems $\dot{x} = A(t)x$ a compact *dichotomy spectrum* (related to the Sacker-Sell spectrum [4]) is established under very general conditions.

2. *Gap conditions* on the dichotomy spectrum for the smoothness class of integral manifolds [2] and foliations are given.

3. Poincaré's non-resonance condition [3] on eigenvalues known from the autonomous case is generalized to a non-resonance condition on compact spectral intervals of the dichotomy spectrum.

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Control of Distributed Autonomous Robotic Systems

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Self-organized and error-resistant control of distributed autonomous robotic units is achieved by using a specifically constructed dynamical system. The robotic units have to be assigned to targets in a manufacturing environment with obstacles in a cost effective way. Besides the dynamic control of the robotic units in space, the underlying optimization problems are the (two-index) assignment problem or the $\mathcal{NP}$-hard three-index assignment problem.

The used differential equations are based, first, on the selection of modes which appears in pattern formation of physical, chemical and biological systems (see e.g. [1] or [2]). Coupled selection equations based on these pattern formation principles can be used as dynamical system approach to assignment problems [3], [4]. The obtained solutions always respect the constraints of the considered assignment problem [3]. Second, a model of Behavioural Forces is used, which has been successfully applied to describe self-organized crowd behaviour of pedestrians [5].

This approach includes collision avoidance as well as error resistivity, i.e., in systems where failures are of concern, the system covers up for sudden external changes like breakdowns of some robotic units. Furthermore, the control guarantees always feasible solutions, i.e., no spurious states cause the system to fail which is of great importance in industrial applications. Computer simulations verify these results and demonstrate the capability of this approach.

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Lyapunov functions method in practical stability

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Let us consider a system of differential equations

$$\frac{dx}{dt} = f\left(x, t\right) \tag{1}$$

where x is a n-dimensional vector, the vector function $f\left(x, t\right)$ satisfies the conditions of the existence and uniqueness theorem, $f\left(0, t\right) \equiv 0$, $t \in [t_0, T]$.

We denote $x\left(t\right) = x\left(t, x_0, t_0\right)$ the trajectory of the system (1) at Cauchy condition $x\left(t_0\right) = x_0$ and also assume that $\Phi_t \subset R^n$ are compacts, $0 \in \Phi_t$, $t \in [t_0, T]$, $G_0 \subseteq \Phi_{t_0}$, $0 \in G_0$.

Definition. The unperturbed solution $x\left(t\right) \equiv 0$ of the system (1) is said to be $\{G_0, \Phi_t, t_0, T\}$-stable if $x\left(t, x_0, t_0\right) \in \Phi_t$, $t \in [t_0, T]$ as soon as $x_0 \in G_0$.

Theorem. For the trivial solution of the system (1) to be $\{G_0, \Phi_t, t_0, T\}$-stable it is necessary and sufficient that there exists a continuous nonincreasing on the system (1) solutions Lyapunov function $V\left(x, t\right)$ such that

$$\{x \in R^n : V\left(x, t\right) \leq 1\} \subseteq \Phi_t, t \in \left[t_0, T\right], \tag{2}$$

$$G_0 \subseteq \{x \in R^n : V\left(x, t_0\right) \leq 1\}. \tag{3}$$

Corollary. If the trivial solution of the system (1) is $\{G_0, \Phi_t, t_0, T\}$-stable and G_0 is compact then the function

$$V\left(x, t\right) = \begin{cases} 1 + \min\limits_{y \in \partial G_0} \rho\left(\varphi\left(t, t_0, x\right), y\right), x \in \{x \in R^n : \varphi\left(t, t_0, x\right) \in R^n \backslash G_0\}, \\ 1 - \min\limits_{y \in \partial G_0} \rho\left(\varphi\left(t, t_0, x\right), y\right), x \in \{x \in R^n : \varphi\left(t, t_0, x\right) \in G_0\}. \end{cases}$$

satisfies all the theorem conditions. Here ρ is a metric equivalent to Euclidean one, $\varphi(t, t_0, x) = x_0$ on the solution $x(t, x_0, t_0)$.

Further, if the set of initial data is starry compact, then it is possible building Lyapunov function which belongs to differentiable functions class. We also estimate the optimal sets of initial conditions in structural forms for linear system and concrete phase constraints [1,2].

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4.3 Dynamics and Algorithms

Organizer : Eusebius Doedel

Key note lecture

Computing Periodic Orbits of Vector Fields

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Periodic orbits are fundamental structures of flows. New methods for computing periodic orbits and their bifurcations with high accuracy are being developed with the use of *automatic differentiation.* This lecture will present these methods, make comparisons with other methods and discuss outstanding problems. Case studies include stiff systems with multiple time scales and the computation of canards.

Invited lectures

Convergence, parameter-identification and C++-code for bifurcation numerics in real life problems

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This talk presents a resume of parts of the results of our *Marburg group.*

In real life problems, the corresponding equations, often PDEs, are known. Numerical computation of bifurcation scenarios is an important task for applications. Results for bifurcation numerics from the last decade, e.g. [4],[5], only cover a few methods, such as specific finite element methods.

We use reaction-diffusion and Navier-Stokes equations as motivating examples. The classical concepts of stability and consistency, yielding convergence, do not apply to the singular situation in bifurcation. So we formulate a new general theory, e.g. [1],[2],[6], modifying these concepts: Stability for bordered systems and the second property, the consistent differentiability, are proved, e.g. for the above examples and for finite difference, finite element and spectral methods. It is particularly important that the stability for bordered systems is the consequence of available properties, e.g. the linearized operator represents a compact perturbation of a monotone operator and for this monotone operator the standard inf-sup-conditions for discretizations are satisfied. Under these conditions the numerical bifurcation scenarios with all the interesting bifurcating solutions, dynamical properties, transformations of the en-foldings back to the original situation, see [2],.. indeed converge to the the original scenarios.

Hence, the huge number of published bifurcation numerics for operator equations, based on bordered systems indeed present converging results: These papers are either directly covered by the above results or can probably be shown to satisfy these conditions.

Based on these results, a C++ code is being developed, such that parameter dependent solution curves can be continued and the bifurcation and dynamical properties be determined. For a new class of problems the appropriate discretization has to be introduced fitting to the side conditions of the code. Then this software yields the above informations for the new problem.

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Numerical bifurcation analysis of delay equations

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In this talk we discuss numerical bifurcation analysis of retarded functional differential equations (RFDEs) with multiple constant delays. Steady state solutions of such equations are independent of the delays and can be found with any standard package for bifurcation analysis of ordinary differential equations. Their stability, however, is determined by the infinite roots of a transcendent characteristic equation. We describe an algorithm which computes the rightmost, i.e. stability determining, roots of this equation. When a steady state solution looses its stability via a Hopf bifurcation a branch of periodic solutions arises. Computing periodic solutions of RFDEs is an infinite dimensional problem because a (periodic) solution is not determined by one point at a given time, instead a function segment with length the maximal delay has to be determined. We discuss numerical computation of the periodic solution and approximation of the dominant Floquet multipliers using a shooting and a collocation approach. We end with a brief discussion of the extra difficulties arising in the study of neutral functional differential equations.

Numerical Investigation of Periodic solutions of multibody systems

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A simple wheelset rolling with constant velocity on a straight track oscillates laterally in a stable periodic motion if its velocity exceeds a certain critical value. The equations of motions of such mechanical multibody systems are differential algebraic equations of index three. Motivated by this application we present numerical methods for the investigation of periodic solutions of multibody systems depending on parameters. A projected collocation method approximates periodic solutions and provides Floquet multipliers, which characterize the stability of an approximated solution. A branch of periodic solutions can be traced using continuation methods, where the Floquet multipliers indicate bifurcations on the branch. Finally, numerical results obtained with a wheelset model are presented.

Ordinary differential equations, differential-algebraic equations and their use in optimization

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We consider the following general smooth optimization problem:

$$\text{Minimize } f(x),\ x \in D \subset \mathbb{R}^N \text{ open, subject to } g(x) = 0 \text{ and } k(x) \geq 0. \tag{1}$$

Our approach to solve (1) is to integrate an ordinary differential equation (ODE) appropriate to (1). With the slack variables y as well as $z = (x, y)$, $\bar{f}(x, y) = f(x)$,

$\bar{g}(x,y) = (g(x), k(x) - \mathrm{diag}(y)y)$, the projector $Q(z) = D\bar{g}(z)^T(D\bar{g}(z)D\bar{g}(z)^T)^{-1}D\bar{g}(z)$ such an equation reads

$$\dot{z} \;=\; (I - Q(z))(-\nabla\bar{f}(z)) - D\bar{g}(z)^T(D\bar{g}(z)D\bar{g}(z)^T)^{-1}\bar{g}(z). \qquad (2)$$

Using a BDF-method to integrate (2) this gives a reliable optimization code, which is particularly well suited for highly nonlinear optimization problems. But in comparison with a classical SQP-method it is not so efficient. This is because one has to solve a linear system of equations to evaluate the right hand side of (2) once. In addition, one is not able to exploit the structures of g and k, since they are destroyed by Q.

To significantly improve the efficiency of that approach, we rewrite (2) as an index 2 differential-algebraic equation (DAE). In this formulation the evaluation of the right hand side is less costly and allows to take advantage of the structures of the constraints. To solve the DAE we use a BDF-code with the speciality that functional iterations instead of Newtons method is used to solve the nonlinear equations in every time step. This substantially decreases the number of evaluations of the first derivatives of f, g and k.

In addition, we discretize the ODE (2) as well as the corresponding DAE with classical one- and linear multistep methods and present discrete convergence results to underpin the power of that approach theoretically.

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Domain on Bifurcation

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The virtual **World of Bifurcation** (WOB) has been established as a www-domain.

WOB combines a database of bifurcation problems with a tutorial on nonlinear phenomena. The name of the domain is `www.bifurcation.de`.

The approach is example-oriented and experimental. The emphasis is on examples that are application-oriented. Most of the examples are dynamical systems.

The first version of WOB has included a set of 12 examples such as a trigger circuit, a reaction in a catalyst particle, Hodgkin-Huxley's nerve model, and voltage collapse in a power system.

It is planned to extend WOB into several directions. Suggestions are welcome. WOB includes figures in postscript format. The figures and other items may be downloaded. Currently large emphasis is placed on developing computer demos that show dynamical systems. A first set of demos is DOS-oriented, a future set of demos will be based on Java.

Contributed talks

Kinetic of growth of Guinier-Preston zones in alloys

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The Guinier-Preston zones directly affect the mechanical and electrical proprieties of metallic alloys. Indeed, for the case of the Al-rich Zn alloys the maximum in resistivity which occurs at a time of the order of minutes for room temperature ageing has been associated with a critical zone size [1]. On the other hand, many studies on Al-Ag single crystals [2] have shown that the GP zones have a faced spherical form for ageing temperatures ranging from 20 to 250C.

In this work, a new mathematical model for analysing the growth of G- P zones in alloys is presented and solved. The new model combines the reaction diffusion equations given by the vacancy pump model [3] and a surface limited mass-transfer boundary condition, which reflects surface-limited mass-transfer between the G-P zones and the matrix. An important result of the new model is the introduction of a new variable, h_m, which is defined as the surface-limited mass-transfer coefficient.

A new implicit finite difference method has been developed for solving the nonlinear Reaction-Diffusion equation [4]. This new approach removes the stability constraints of the explicit treatments presented in the literature and allows one to use considerably larger time steps. The numerical program developed gives the radius temporal variation R(t) of the G-P zones, the obtained results present a good agreement with experimental resistivity curves obtained in Al-Zn alloys [5], for relatively higher times the radius decreases after a critical value R_0 . The influence of the other parameters on the critical size of G-P zones, for example the aging and quenched temperatures, is also studied. Radial concentration of different species and the total concentration decay could also be provided.

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A numerical method for computing unstable quasi–periodic solutions for the 2–D Poiseuille flow

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It is known that the laminar flow in a plane channel becomes unstable to small disturbances for Reynolds number over the critical one, $Re_{\rm cr} = 5772.22$, for $\alpha = 1.02056$, being α the wavenumber. In this work we compute, for several values of α, the branch of unstable solutions which bifurcate from the laminar one. These solutions are periodic on time. Due to the translational symmetry of the channel, they behave as rotating waves, which are steady in the frame of reference moving with the wave speed c. In this way we can find stable and unstable rotating waves, whose stability is examined through the spectrum of the corresponding linearization in the Navier–Stokes equations.

This analysis allow us to confirm previous results with regard to the existence of two Hopf bifurcations for $\alpha = 1.02056$ in the Reynolds–amplitude curve, namely when $Re \approx 4700$ and $Re \approx 7400$. We also verify that the first bifurcation occurs further from the minimum Reynolds of the curve (≈ 4660). From these bifurcations emanates a family of quasi–periodic solutions (modulated waves) which, again due to symmetry, may be viewed as periodic ones when the observer is moving with an appropriate speed. Using techniques from finite–dimensional dynamical systems, we follow, by means of a continuation method, one branch of quasi–periodic orbits. Newton's method, applied to search for fixed points of a suitable Poincaré map of the flow, allows us to find stable and unstable modulated waves.

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Rigorous discretization of subdivision techniques

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Recently efficient set-oriented methods have been proposed for the numerical investigation of dynamical systems [1,2,3]. A basic question arising in implementing these methods is to compute the "set-wise image" of some set: determine all sets of some collection which intersect the image of the given set. In this talk we describe how this discretization question can be tackled rigourosly and furthermore how the numerical effort can be reduced to a minimum. As an example we compute rigorous coverings of stable and unstable manifolds in the Henon map and show that they intersect transversally. We indicate how these methods carry over to ordinary differential equations.

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Two-dimensional global manifolds of vector fields

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We present an algorithm for computing two-dimensional stable and unstable manifolds of three-dimensional vector fields. The main idea is to grow the manifold in concentric (topological) circles. Each new circle is computed as a set of intersection points of the manifold with a finite number of planes perpendicular to the last circle. Together with a scheme for adding or removing such planes this guarantees the quality of the mesh representing the computed manifold. As examples we show the stable manifold of the origin spiralling into the Lorenz atractor, and an unstable manifold in Arneodo's system converging to a limit cycle.

Strong singularities and the continuous Newton method

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The formulation of continuous-time analogues of iterative methods for root-finding and optimization problems can be traced back to Davidenko in 1953, and found a great development in the decade of 1970, with several papers by Abbott and Brent, Boggs, Branin and Smale, among others (see references in [1,3]). Since then, continuous models have been further developed in the context of homotopy techniques and, more specifically, as *trajectory methods* [1]. In the continuous-time setting, a unique model may lead to different iterative techniques, including damped and accelerated methods, through different integration schemes. This approach shifts the convergence analysis of these iterations to a stability study of the continuous system and the discretization method. Continuous models display a better global behavior and, specifically, the difficulties arising when trajectories of Newton-based methods approach singular points in the search of regular roots can be overcome.

The existence of singular roots introduces additional problems. In the context of the discrete Newton method, the basic theorem, due to Reddien [5], states linear convergence when approaching a singular zero from a cone-shaped region with vertex in the root. This result has been later extended in several directions, mainly by Decker and Kelley, Griewank and Keller (see [2,4] and references therein).

In this talk, the local behavior of the continuous-time analogue of Newton method is studied at singular roots x^* of $f \in C^3(\mathbb{R}^n, \mathbb{R}^n)$ through a taxonomy of these into *weak* and *strong* ones and assuming $\nabla \det f'(x^*) \neq 0$. The weak case, defined by the condition $f(x) \in \mathrm{Im} f'(x)$ for all x in the local singular manifold around x^*, characterizes both discrete and continuous-time situations in which a singular root has a spherical attraction domain including singularities, against a common assumption in this context [2]. A theorem on directional stability in the generic case of strong singular roots is then proven via a Lyapunov-Schmidt approach. Forward Euler discretization yields Reddien's result.

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Posters

Analitycal Formulae for Solutions of Difference Equations

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This paper concerns the analytical formulae for solutions of linear difference equations with non-constant coefficients. We present analitycal formulae which are non-recurrent algorithms for obtaining solutions of equations. We use some kind of sum operators and elements of the graph theory.

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4.4 Exponentially Small Phenomena

Organizer : Carles Simó

Key note lecture

Analytical and numerical detection of exponentially small phenomena

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Exponentially small phenomena occur in a wide variety of problems. Always dealing with problems which are analytical, we find these phenomena both in estimates of the remainders in normal forms of vector fields and diffeomorphisms and in averaging methods and, in general, in most of the problems which cannot be detected by using any finite order approach of the classical perturbation theory. They can be revealed by a suitable use of complex variable, by extending the phase space and time to suitable complex strips. They also appear in phenomena like the delay of the bifurcation for fixed points and periodic solutions, in the case of slowly varying parameters, in adiabatic invariance and in the corresponding bifurcation diagrams of all these problems.

We describe different methods to detect this exponential smallness, dealing both with analytical and numerical aspects. In some problems, the use of Melnikov method can give a key to predict the correct order of magnitude. However, some examples will be displayed where a direct use of Melnikov method fails to give the correct estimate. This is specially true in systems involving several fast frequencies. In that case, the correct answer depends strongly on the arithmetic properties of the frequencies, and some bifurcations are induced by the arithmetics.

Numerically it can be a hard task to detect these phenomena. They can be revealed and estimated by going to the complex. Keeping the computations in the reals requires high precision arithmetics.

Invited lectures

Singular separatrix splitting and Melnikov theory for nearly-integrable systems

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Several results are presented, based on [1], [2], [3], about the applicability of the *Melnikov potential* to the detection of the exponentially small splitting of separatrices that takes place in some families of nearly-integrable exact symplectic maps and Hamiltonian flows.

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Exponentially small splitting of separatrices near bifurcations in area-preserving maps

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When the saddle-center bifurcation occurs in an analytic family of area-preserving maps, first a parabolic fixed point appears at the origin and then this point bifurcates, creating an elliptic and hyperbolic fixed point. Separatrices of the hyperbolic fixed point form a small loop around the elliptic point. In general the separatrices intersect transversely and the splitting is exponentially small with respect to the perturbation parameter. We derive an asymptotic formula, which describes the splitting, and study the properties of the preexponential factor.

On accuracy of adiabatic invariant conservation in one-frequency systems

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Consider a Hamiltonian system with one degree of freedom depending on a slowly varying in time parameter. Let for every frozen value of this parameter the phase portrait of the system contain a domain filled with closed trajectories. Let I be the "action" variable in this domain. Suppose that as time tends to $\pm\infty$ the value of the parameter tends to definite limits in such a manner, that the value of the "action" along a trajectory also tends to definite limits $I_{\pm}$. The difference $\Delta I = I_+ - I_-$ is called an accuracy of adiabatic invariant conservation [1]. If the Hamiltonian of the system is an analytic function of its arguments, and the parameter is an analytic function of the "slow time" εt, $0 < \varepsilon \ll 1$, then the value ΔI is exponentially small: $\Delta I = O(\exp(-c/\varepsilon)), c = \text{const} > 0$. In [2] the method for the analytic continuation of solutions of the system under consideration is suggested and an estimate from below for the constant c is found. This estimate can not be improved. In the talk the proof of this estimate on the base of a modification of the method of [2] is given.

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A new method for measuring the splitting of invariant manifolds

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We study the so-called Generalized Arnold Model (a weakly hyperbolic near-integrable Hamiltonian system), with $d+1$ degrees of freedom ($d \geq 2$), in the case where the perturbative term does not affect a fixed invariant d-dimensional torus. This torus is thus independent of the two perturbation parameters which are denoted ε ($\varepsilon > 0$) and μ.

We describe its stable and unstable manifolds by solutions of the Hamilton-Jacobi equation for which we obtain a large enough domain of analyticity. The splitting of the manifolds is measured by the partial derivatives of the difference ΔS of the solutions, for which we obtain upper bounds which are exponentially small with respect to ε.

A crucial tool of the method is a *characteristic vector field*, which is defined on a part of the configuration space, which acts by zero on the function ΔS and which has constant coefficients in well-chosen coordinates.

It is in the case where $|\mu|$ is bounded by some positive power of ε that the most precise results are obtained. In a particular case with three degrees of freedom, the method leads also to lower bounds for the splitting.

Averaging in multi-frequency slow-fast systems

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We consider the multi-frequency slow-fast system

$$\dot{x} = \omega + \varepsilon f(x, y, \varepsilon), \quad \dot{y} = \varepsilon g(x, y, \varepsilon).$$

Here $x \in \mathbf{T}^n$, $y \in \mathbf{R}^m$, ε is a small parmeter, the constant vector $\omega \in \mathbf{R}^n$ is Diophantine, and the functions f, g are real-analytic. It is well-known that by a change of the variables $X = x + \varepsilon\xi(x, y, \varepsilon), Y = y + \varepsilon\eta(x, y, \varepsilon)$ the system can be reduced to the form

$$\dot{X} = \omega + \varepsilon F_0(y, \varepsilon) + \varepsilon F_1(x, y, \varepsilon), \quad \dot{Y} = \varepsilon G_0(y, \varepsilon). + \varepsilon G_1(x, y, \varepsilon),$$

with F_1 and G_1 exponentially small with respect to ε. The functions F_1, G_1 depend on the change, but they can not be removed in general.

We are interested in estimates for the "smallest possible" F_1 and G_1.

Contributed talks

Time averaging of parabolic partial differential equations: exponential estimates

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We consider a class of systems of nonautonomous parabolic partial differential equations, where the time dependence is a small rapid forcing

$$\begin{aligned} \frac{\partial U}{\partial t} &= \operatorname{diag}(d_1, \ldots, d_n)\Delta U + F(U) + hG(U, \frac{t}{h}, h) \\ U(0) &= U_0 \in H^s_{per}(\Omega, \mathbb{R}^n) \end{aligned}$$

We assume periodic boundary conditions on $\Omega = [0, l]^d$, enough smoothness on the initial conditions by $s > d/2$ and suitable hypotheses of analyticity on the local nonlinearities F and G. Then we can show in [2] a counterpart for parabolic partial differential equations of the well known result by Neishtadt [3] for ordinary differntial equations. By an analytic time dependent coordinate change we get a new equation:

$$\begin{aligned} \frac{\partial V}{\partial t} &= \operatorname{diag}(d_1, \ldots, d_n)\Delta V + F(V) + \tilde{F}(V) + h\alpha(V, \frac{t}{h}, h) \\ V(0) &= U_0 \in H^s_{per}(\Omega, \mathbb{R}^n), \end{aligned}$$

where the nonautonomous remainder α is exponentially small after any time $t > 0$ with a bound of the form $\exp(-\min(c, t)h^{-\frac{1}{3}})$. The correction terms $\tilde{F}$ and the remainder α are nonlocal in space. The proof is based on Galerkin approximation, an adaption of Neishtadt's methods on finite dimensional approximation space and the use of very high regularity. Following Ferrari and Titi one can show, that the solutions $V(t)$ are in Gevrey classes $G^{s/2}_\sigma = D((-\Delta)^{s/2} \exp(\sigma(-\Delta)^{\frac{1}{2}}))$. For positive times the solutions are analytic in the spatial variable and the Fourier modes decay exponentially fast. Applications can be made to the splitting of the homoclinic orbits.

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4.5 Geometric Integrators

Organizer : Jerrold E. Marsden

Key note lecture

Multisymplectic Integrators

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For first order Lagrangian field theories, the existence of the fundamental geometric structures as well as their preservation along solutions can be obtained directly from the variational principle. Using this observation, we constructed in [1] integrators for Lagrangian field theories and identified the structures they preserved, by direct discretiztion of the variational principle.

My talk will be a review of this work.

References

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Invited lectures

Variational Structure/Numerics of Contact Problems

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The purpose of this talk is to analyze a variational formulation of problems involving contact and friction. Some numerical examples are done using the Coulomb model of friction to illustrate the method.

Geometric Annealing

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The author discusses dynamical approaches to global minimization and suggests an alternative to simulated annealing based on the combination of a temperature-controlled extended Hamiltonian with efficient symplectic integrators.

The problem of finding the global minimum or nearly-global minima of a smooth nonlinear function f of several variables—possibly with respect to one or more constraints—is widespread and fundamental to scientific and engineering research, including, in particular, problems in atomic and molecular physics, design of materials, protein structure, and sphere packing. The most popular methods share certain obvious features: (1) they are iterative in nature, generally requiring a large number of steps to find the global minimum of a complex function, (2) they rely on successive evaluations of f and—often—its gradient to direct the search process, and (3) in the process of approaching the minimum they allow the value of f to increase as well as decrease, in order to surmount local barriers.

Dynamical approaches to the global minimum have been developed based on a gradient flow or a damped Hamiltonian flow. In either model, the local minima of f become exponentially stable equilibria; for this reason, these flows may converge too rapidly to local minima to form the basis of a global minimization procedure. (Although, in the context of mild damping, it is interesting to note that certain numerical discretization methods appropriately damp the symplectic area form.)

In this talk, an alternative approach to global minimization dynamics is developed based on the combination of Nosé's thermostated Hamiltonian with a cooling schedule (as in 'simulated annealing'). Alternative formulations of the Nosé temperature control are derived, with an emphasis on facilitating efficient symplectic integration. The numerical stability of various approaches with respect to variation of the temperature is investigated, and results of experiments with model problems are presented.

Symmetry, pseudospectral methods, and conservative PDEs

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Preserving the structure, for example the symmetry, of continuum objects under discretization has always been part of the field of numerical PDEs. It is well known, for example, that Fourier pseudospectral differentiation matrices are antisymmetric while Chebyshev ones are not. This is intimately related to the linear and nonlinear stability of methods which used these matrices. In this talk I discuss this situation in detail and show how the Chebyshev matrices *are*, in fact, skew-adjoint with respect to an appropriate inner product, and how they can therefore provide conservative discretizations of some wave equations, with corresponding nonlinear stability and Hamiltonian structure. For more complicated (e.g. Euler) equations, normal pseudospectral methods are not conservative, but with care—discretizing the conserved energy and the equation compatibly, and using antialiasing and fast transforms—they can be made to be so.

Symmetry Reduction of Discrete Lagrangian Mechanics on Lie groups

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We show that when a discrete Lagrangian $L : G \times G \to \mathbb{R}$ is G-invariant, a Poisson structure on (a subset) of one copy of the Lie group G can be defined by means of reduction under the symmetry group G of the canonical discrete Lagrange 2-form ω_L on $G \times G$. Alternatively, for the corresponding reduced discrete mechanical system on a Lie group G determined by the (reduced) Lagrangian ℓ we can define a Poisson structure via the pull-back of the Lie-Poisson structure on the Lie algebra g^* by the corresponding Legendre transform. Our main result shows that these two structures coincide and govern the corresponding discrete reduced dynamics. In particular, the symplectic leaves of this structure become dynamically invariant manifolds which are manifestly preserved under the structure preserving discrete Euler-Poincaré algorithm.

Moreover, starting with a discrete Euler-Poincaré system on G one can readily recover, by means of the Legendre transformation, the corresponding Lie-Poisson Hamilton-Jacobi system on g^* analyzed by Ge and Marsden [1]; the relationship between the discrete Euler-Lagrange and discrete Euler-Poincaré equations and the Lie-Poisson Hamilton-Jacobi equations was examined from a different point of view in [2].

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Making Waves: Multi-Symplectic Methods for Hamiltonian PDEs

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Various Hamiltonian PDEs can be reformulated as a multi-symplectic PDEs which implies the existence of a local conservation law of symplecticity. We take this conservation law as the starting point for deriving multi-symplectic methods for PDEs in a similar manner as done in case of symplectic methods for Hamiltonian ODEs. We will discuss multi-symplectic schemes based on Runge-Kutta collocation and finite volume methods. Numerical results will be presented for the sine-Gordon equation and a simplified shallow water system.

Contributed talks

Finite-Difference Approximations and Cosymmetry Conservation in Filtration Convection Problem

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We study different conservative finite-difference discretizations of partial differential equations (PDEs) with respect to preservation of cosymmetry property. The concept of cosymmetry was introduced recently by Yudovich [1], [2], and some interesting phenomena was found for both ordinary differential equations (ODEs) and PDEs. It was shown that cosymmetry may be a reason for the existence of continuous family of regimes of the same type and this family is not connected with a symmetry group. Symmetry implies identical spectrum for all points on the family, while the stability spectrum for cosymmetric systems depends on the location of a point. We consider two-dimensional planar Darcy filtration convection of saturated incompressible viscous fluid in a rectangular container filled with porous medium. We are interested in accurate computation of the family of equilibria. The key point here is discretization of nonlinear terms of the PDE, which are represented by the Jacobian. Several finite difference approximations of Jacobian are compared and it was found that the Arakawa scheme provides the most accurate results due to its conservation properties. The numerical results given demonstrate that conservative finite-difference methods can be effectively used to compute the continuous family of stationary regimes. We give some evidence of family degeneration when unappropriate approximations were used, which do not preserve the skew-symmetry of the Jacobian and nullification of the gyroscopic forces.

References

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Equivalence of Newmark and Variational Algorithms and Extensions to Dissipative Mechanical Systems

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We present two related results. First, for conservative systems, we offer an explanation for the unusually good performance of the classical Newmark family of integrators.

This is done by showing that these algorithms, as well as related schemes, are variational in the sense of the Veselov discrete mechanics. Such variational algorithms are examples of symplectic–momentum preserving integrators and exhibit the excellent global energy behavior typical of geometric methods. Analytical results concerning variational integrators can, through the established equivalence, be applied to explain the observed numerical behavior of the Newmark algorithm.

Second, we extend the variational framework to include forced mechanical systems and, in particular, those with dissipation. This is done both by direct discretization of the Lagrange-d'Alembert principle and by the use of a variational formulation of dissipation. We demonstrate that integrators derived in this manner have good numerical behavior in that they correctly estimate the total change in energy and momenta over the integration run. This extends some of the advantages characteristic of geometric integrators for conservative systems to the dissipative realm.

Posters

Constant Temperature Molecular Dynamics using an Extended Lagrangian Method

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We present a new extended phase space method for constant temperature (canonical ensemble) molecular dynamics. Our starting point is the extended Hamiltonian introduced by Nosé to generate trajectories corresponding to configurations in the canonical ensemble. Using a Poincaré time-transformation, we construct a Hamiltonian system with the correct intrinsic timescale and show that it generates trajectories in the canonical ensemble. Our approach corrects a serious deficiency of the standard change of variables (Nosé-Hoover dynamics), which yields a time-reversible system but simultaneously destroys the Hamiltonian structure. A symplectic discretization method is presented for solving the Nosé-Poincaré equations. The method is explicit and preserves the time-reversal symmetry. Extensions are presented for classical spin systems.

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4.6 Numerical Ergodic Theory

Organizer : Michael Dellnitz

Key note lecture

Set Oriented Numerical Methods for Dynamical Systems

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Frequently dynamical systems exhibit complicated temporal behavior. In this case it can be useful to approximate corresponding characterizing statistical quantities such as Lyapunov exponents or invariant measures. For this purpose there have been proposed reliable numerical techniques, which are based on a set-oriented approach rather than on long term simulations of the underlying dynamical system. Here we give an overview of recent developments in this area.

In particular, we explain how to approximate a natural invariant measure, that is, an *SRB-measure* if such a measure exists. These measures provide the information about the frequency by which a typical solution is observed in different parts of state space. The numerical approximation is obtained by adaptive multilevel subdivision strategies based on box coverings of the corresponding invariant set. (Once the invariant measure is known Lyapunov exponents can efficiently be computed by spatial integration, see [1].)

In addition to the stationary statistical behavior obtained by the invariant measure we show how to approximate *almost invariant sets* (cf. [3]). These are regions in state space where typical solutions stay for a relatively long period of time before leaving again. (The concept of *almost invariance* has turned out to be useful in the approximation of so-called *conformations* of molecules of moderate size, see [4].)

There are several convergence results for the (adaptive) numerical methods which are presented here, e.g. the convergence of approximating measures to an SRB-measure. Additionally we justify the techniques for the computation of almost invariant sets by recent analytical results concerning the spectrum of the *Perron-Frobenius operator* (cf. [2]). We illustrate both analytical results and numerical methods by several examples.

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Invited lectures

Numerical Problems for Random Dynamical Systems

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We will review the state of the art of the numerical approximation of invariant objects of random dynamical systems, in particular of

– Lyapunov exponents and rotation numbers,
– random point attractors,
– random set attractors,
– invariant manifolds,
– stationary and invariant measures,
– entropy.

Markov modelling of random dynamical systems

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Techniques for estimating the stationary distribution of deterministic systems based on a discrete Markov approximation of the dynamics are well known and have been successfully used in the past. We now extend these techniques to random dynamical systems by defining a suitably averaged Markov model. We find that these constructions are often superior to iterative orbit based methods and for some classes of maps provide rigorous error bounds for our estimates of the stationary distribution.

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Finite Representation of the Conley Decomposition

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A finite precision representation of the Conley decomposition of a continuous map acting on a compact space is presented providing a method to find chain transitive sets and isolating neighborhoods in terms of communication classes of a Markov chain associated with a partition of phase space. When an attracting set supporting an SBR measure exists we can prove the convergence of a set approximation of the attractor and show the relation between the invariant measure and the stationary measure of the chain. When the invariant set is Lyapounov stable and transitive, the work of Buescu and Stewart show that the measure is uniquely ergodic. We use this result to show converence of an approximate measure to the SBR measure.

Contributed talks

Numerical Schemes for Stochastic Delay Differential Equations

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We are concerned here with stochastic delay differential equations or *SDDEs*, which generalise both deterministic delay differential equations (DDEs) and stochastic ordinary differential equations (SODEs). This class of problems is of practical interest, e.g. in population dynamics and robotics.

Let $(\Omega, \mathcal{A}, P)$ be a complete probability space with a filtration $(\mathcal{A}_t)$ satisfying the usual conditions; i.e. the filtration $(\mathcal{A}_t)_{t\geq 0}$ is right-continuous, and each $\mathcal{A}_t$, $t \geq 0$, contains all P-null sets in $\mathcal{A}$. Let $W(t)$ be the 1-dimensional Brownian motion given on this space. With $\tau > 0$ we denote by $C := C([-\tau, 0], \mathbb{R})$ the Banach space of all continuous paths $\eta : [-\tau, 0] \to \mathbb{R}$ given the supremum norm $\|\eta\|_C := \sup_{s\in[-\tau,0]} |\eta(s)|$, $\eta \in C$. Let $0 = t_0 < T < \infty$ and consider the stochastic delay differential equation (SDDE) with one fixed lag τ:

$$\left.\begin{array}{rcl} dX(t) &=& f(t, X(t), X(t-\tau))\ dt\ +\ g(t, X(t))dW(t), \quad t \in [0, T] \\ X(t) &=& \Psi(t), \quad t \in [-\tau, 0] \end{array}\right\} \qquad (1)$$

where $\Psi(t)$ is an $\mathcal{A}_{t_0}$-measurable $C([-\tau, 0], \mathbb{R})$ valued random variable such that $\mathcal{E}\|\Psi\|^2 < \infty$. The SDDE (1) is interpreted in the Itô sense. If g does not depend on X, the equation has *additive noise*, otherwise it has *multiplicative noise*. An $\mathbb{R}$-valued stochastic process $X(t) : [-\tau, T] \times \Omega \to \mathbb{R}$ is called a *strong solution* of (1), if it is a measurable, sample-continuous process such that $X|_{[0,T]}$ is $(\mathcal{A}_t)_{0\leq t\leq T}$-adapted, $f : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $g : [0, T] \times \mathbb{R} \to \mathbb{R}$ are continuous functions and X satisfies (1) almost surely. It is well known, that a unique strong solution to Equation (1) exists, if f and g satisfy Lipschitz conditions and a linear growth bounds.

We address the principal issues involved in progressing to the numerical treatment of Equation (1) and our purpose is to ascertain how familiarity with methods used for DDEs and for SODEs can be exploited to yield methods for approximating strong solutions of SDDEs. We will present numerical results. The work is jointly performed with Christopher Baker, at Manchester.

Supported by EU grant ERBFMBICT983282

Young Measures and Invariance

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Invariant measures are an important tool in discussing the statistics of long-term behaviour of dynamical systems. There are two difficulties associated with this approach to dynamics. One is that for discontinuous systems, which are becoming more and more important in applications, the set of invariant measures may prove to be empty. Another problem is that the set of invariant measures may prove to be too big in a sense, and there arises the problem of picking a "physical" invariant measure from this set. In this presentation we make a case for the use of **Young measures**, introduced initially by L. C. Young in the context of chattering controls. Examples of analytic and numerical computation of Young measures for many well-known dynamical systems will be given.

Calculation of the Morse Spectrum of a Dynamical System

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A discrete dynamical system generated by a homeomorphism $\mathbf{F}$ defined on a vector bundle $\mathbf{E}$ and linear on fibers is considered. The investigation is motivated by the inspection of Lyapunov exponents of a differential on the tangent space. The Morse spectrum is a limit set of Lyapunov exponents of periodic ε-trajectories as $\varepsilon \to \mathbf{0}$. Our aim is the calculation of the Morse spectrum without any preliminary information on a dynamical system. The investigation is based on methods of symbolic dynamics and all estimates required can be obtained by standard numerical methods. A common scheme of the study is the following. By using a covering of the associated projective bundle $\mathbf{PE}$ by cells, a dynamical system is associated with the directed graph $\mathbf{G}$ called a symbolic image. The symbolic image $\mathbf{G}$ can be considered as a finite discrete approximation of the associated mapping $\mathbf{PF}$ on the projective bundle. A lot of the effective information on the global structure of a system on the projective bundle can be obtained by the analysis of the symbolic image. Periodic paths on the symbolic image generate the set of Lyapunov exponents that are estimates of Lyapunov exponents of periodic ε-trajectories. The Morse spectrum of the symbolic image is defined as a limit set of Lyapunov exponents of periodic paths. It turns out that the Morse spectrum of a symbolic image is determined by Lyapunov exponents of simple periodic paths. A periodic path is simple if it passes through different vertices. The set of simple periodic paths is finite. This allows one to calculate the Morse spectrum of a symbolic image. The Morse spectrum of a symbolic image is in turn an estimate of the Morse spectrum of the initial mapping $\mathbf{F}$ depending on diameters of covering cells. In this manner a neighborhood of the Morse spectrum of $\mathbf{F}$ can be obtained. By applying a subdivision on the covering, a sequence of embedded neighborhoods of the Morse spectrum converging to the Morse spectrum of $\mathbf{F}$ is constructed. The method discussed above is realized as a computer software. The localization of the Morse spectrum allows one to verify hyperbolicity by mean of a computer.

The research is supported by the Russian Foundation for Basic Researches under Grant 97-07-90088.

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4.7 Numerics of Dynamics

Organizer : Celso Grebogi

Key note lecture

Shadowability of Chaotic Dynamical Systems

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In studying their systems, physical scientists write differential equations derived from fundamental laws. These equations are then used to understand, analyze, predict, and control the system's behavior, provided one is able to determine the solutions. As the role of nonlinearity grows in importance for the study of physics, solutions often cannot be obtained in closed form, and numerical solutions must be relied on. Computers are now an integral part of the physicist's *modus operandi*.

A basic question always present when obtaining numerical solutions is to what extent they are valid. This question is especially meaningful when dealing with chaotic dynamics, since local sensitivity to small errors is the hallmark of a chaotic system. Floating-point calculations commonly used to approximate solutions of differential equations or compute discrete maps produce *pseudo-trajectories*, which differ from true trajectories by new, small errors at each computational step. Despite the sensitive dependence on initial conditions, the methods of shadowing have shown that for chaotic systems that are hyperbolic or nearly hyperbolic, locally sensitive trajectories are often *globally insensitive*, in that there exist true trajectories with adjusted initial conditions, called shadowing trajectories, very close to long computer-generated pseudo-trajectories. A dynamical system is hyperbolic if phase space can be spanned locally by a fixed number of independent stable and unstable directions which are consistent under the operation of the dynamics.

In the absence of hyperbolic structure, much less is known about the validity of long computer simulations. Recently it was shown that trajectories of a chaotic system with a fluctuating number of positive finite-time Lyapunov exponents fail to have long shadowing trajectories. In other words, they are globally sensitive to small errors. Such hyperchaotic system has two positive Lyapunov exponents, although finite-time approximations of the smaller of the two fluctuate about zero, due to visits of the trajectory to regions of the attractor with a varying number of stable and unstable directions. The destruction of hyperbolicity caused by this phenomenon leads to global sensitivity – only relatively short pseudo-trajectories will be approximately matched by true system trajectories.

Our discussion of the global sensitivity of trajectories for these non-hyperbolic systems is limited in this talk to the comparison between physical models and computer simulations, but the same questions arise whenever comparing the time behavior of two systems evolving under similar, but slightly different dynamical rules. For example, a natural system and its theoretical *model* differ by modeling errors. In the presence of fluctuating Lyapunov exponents, global sensitivity may lead to trajectory mismatch, in particular when long times are considered. The result is that no trajectory of the theoretical model matches, even approximately, the true system outcome over long time spans.

Invited lectures

Numerical exploration of homoclinic dynamics

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In many instances chaotic behavior in discrete dynamical systems is related to homoclinic and heteroclinic intersections. In this talk we discuss these phenomena from the viewpoint of numerical bifurcation analysis.

First we report on some recent results of J.-M. Kleinkauf [3],[4] on the numerical computation of homoclinic and heteroclinic tangencies in discrete systems. His analysis starts with the observation that these tangencies can be characterized as turnig points of branches of corresponding orbits in sequence spaces. It is then shown how heteroclinic tangencies can be computed numerically in a robust way by solving appropriate finite boundary value problems. Error estimates for the branches are derived including a superconvergence phenomenon for the critial parameter value at which the tangency occurs.

Using these tools we then analyze bifurcation phenomena created through homoclinic tangencies and - more generally - through closed loops of homoclinic and heteroclinic orbits. In a certain sense a homoclinic tangency may be considered as a bifurcation of infinite codimension. We show how this affects the continuation of multi-humped homoclinics and their tangencies and makes it increasingly more difficult to follow them in a reliable way. The above techniques will also be used to extend the methods for evaluating the symbolic dynamics that is created by homoclinic intersections [1]. Finally, we will shed some new light on the well-known result by Fiedler and Scheurle [2] that the discretization of homoclinic orbits generically creates transversal intersections and that these occur in an exponentially small wedge under analyticity assumptions.

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The Computation of Lyapunov Exponents via Spatial Integration

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We propose a new method for the numerical approximation of the largest Lyapunov exponent. This method is based on the computation of a spatial average with respect to an underlying (natural) invariant measure rather than on a long-term simulation of the dynamical system. Thus, this approach can be interpreted as a generalization of the classical power method for eigenvalue problems. We will show that it is particularly advantageous for the detection of so-called *blowout bifurcations* of a synchronous chaotic state, and we illustrate this fact for a system of two coupled Duffing oscillators.

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Modeling and complexity of coupled chaotic systems

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Coupled oscillators are relevant to a large variety of physical and biological phenomena. Arrays of Josephson junctions and coupled solid state lasers are well known examples in physics and electrical engineering. In biology, vital organs such as hearts, auditory, visual, and central nervous systems are complex networks of many small oscillators such as cells and neurons. Coupled equations can also arise from spatial discretization of nonlinear partial differential equations such as the Navier-Stokes equation in fluid dynamics. The dynamics of the fundamental elements, or the individual oscillators in the network, can be either regular or chaotic. Typically, the collective behavior of all the oscillators in the network can be extremely rich, ranging from steady state or periodic oscillations to chaotic or turbulent motions.

The focus of this talk is on modeling and complexity of coupled chaotic oscillators. In particular, it has been known that chaotic dynamics may impose severe limits to deterministic modeling by dynamical equations of natural systems. The obstruction to deterministic modeling is caused by a type of nonhyperbolic behavior in chaotic systems: unstable dimension variability. Roughly, unstable dimension variability means that periodic orbits embedded in the chaotic invariant set have distinct number of unstable directions. Theoretical argument and numerical computations will be presented to show that unstable dimension variability is common in systems of coupled chaotic oscillators. Evidence will also be given indicating that unstable dimension variability may be responsible for complexity observed in these systems.

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Shadowing time for chaotic trajectories

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We explore the statistical aspects of finite-time shadowing in nonhyperbolic dynamical systems. A diffusion model parametrized by information from the finite-time Lyapunov exponent distributions of the system can be used to calculate expected shadowing times and shadowing distances. This explains the mechanism by which long-term shadowing breaks down in cases of "unstable dimension variability", when finite-time Lyapunov exponents fluctuate about zero.

Contributed talks

Dimension and stability of sets embedded in normally hyperbolic invariant manifolds

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We investigate discrete-time dynamical systems on Riemannian manifolds which are generated by smooth maps. Using properties of associated full variational systems and partial variational systems in direction transversal to the considered invariant submanifolds we obtain conditions for orbital stability and instability of such invariant sets. A known stabiliy criterion of Ruelle for periodic orbits of a discrete-time system with invariant measure is formulated in terms of Lyapunov exponents. But in application to concrete systems in synchronization theory such a criterion is not always effective. These difficulties are handled in the present contribution by deriving stability conditions using singular values of the tangent map. A number of these conditions leads to upper or lower estimates of the largest two singular values of the linearized map. The techniques which are developed here are similar to the constructions for continuous-time systems on Riemannian manifolds used in a joint work with G. A. Leonov and A. Noack [1].

Assuming that the discrete-time system is given in feedback control form (see [2]) on the cylinder the conditions for the stability of invariant sets are split into requirements for the linear part (frequency domain conditions) and the nonlinearity. The contribution is also concerned with upper bounds for the Hausdorff dimension of normally hyperbolic invariant sets. The dimension bounds are formulated in terms of singular values of the restricted tangent maps acting in transversal or longitudinal directions [3].

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Verification of chaos in the planar restricted 3–body problem

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We present a computer–assisted method to verify the existence of chaotic behaviour in discrete dynamical systems. Our approach is closely related to the procedure introduced in Stoffer, Palmer [1]. Yet the scheme of [1] is substantially improved leading to a more efficient procedure with a wider range of potential applications. Moreover, the application to the planar restricted three body problem with two primaries of equal masses is offered. As is well known, this problem is described by a four–dimensional system of differential equations. Restricting the flow to an energy surface and defining an appropriate Poincaré section the problem is reduced to a map in $\mathbb{R}^2$ to which our method is shown to apply.

Strictly speaking we do not rigorously establish chaotic behaviour. This is due to the fact that we replace certain rigorous error bounds by what we call realistic estimated upper bounds.

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Wavelet Approach to Nonlinear Dynamics

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We consider a number of dynamical problems which are described by the systems of ordinary differential equations with polynomial nonlinearities and with or without some constraints. We consider variational wavelet approach for constructing explicit solutions of these problems. We have the solutions as a multiresolution expansion in the base of compactly supported wavelets or wavelet packet bases. In the general case we consider biorthogonal wavelet expansions. These solutions are parameterized by solutions of reduced algebraic problems. We consider applications to optimal control problem and nonlinear accelerator physics problems. Also we present applications to a number of Hamiltonian problems and their perturbations. We consider dynamical problems in invariant variational approach via coadjoint orbit picture, semiproducts and metaplectic structure. We construct symplectic, Poisson and quasicomplex structures using generalized wavelets and non-standard (maximum sparse) representations for operators in different coherent, well localized wavelet bases in functional spaces or scale of spaces. We consider applications of our approach to the theory of homoclinic chaos and quasiclassics. Also we consider applications to a number of wave motion problems (nonlinear wave equations) and turbulence problems (Kuramoto–Sivashinsky equations). We consider variational approach for constructing wavelet–Galerkin representations via reduction from initial complicated problems to a number of standard algebraic problems. As a result we obtained explicit representation for well localized coherent structures.

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Remarks on computational aspects of computer assisted proofs of chaotic behavior for ODE's

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I will discuss the rigorous numerics using the computer assisted proofs of chaotic behavior for Lorenz system and Rössler equations [1–3].

The computer part in the above mentioned proofs can be reduced to the following problem:

Compute fast and rigorously with a-priori prescribed error the image of some relatively big rectangle under the Poincaré map for a given ODE.

It should be stressed that we allow for relatively big errors (for example 10% of the size of the important sets in the problem).

In principle this is an easy problem, we just have to discretize both in time (i.e. we use Runge–Kutta , or Taylor method) and in space and then perform finite computations. In the problems mentioned above it required up to 60 hours of computation time to complete the task.

We discuss how the following issues influence the computation time

- the order of the numerical method
- various norms for computation Lipschitz constant of the flow
- Lohner algorithm for error propagation

We show that different approaches lead to drastically different computation times covering range from just minutes to years for the same problem.

The conclusion from these experiments can be formulated as follows:

The most important factor in such computations is propagation of errors due to space discretization. It is reasonable to apply an complicated algorithm (this means that to perform a single time step takes much computer time) if this leads to a significant reduction of the estimate for the Lipschitz constant of the flow.

For example the use of Lohner algorithm in Rössler system can speed up computation by a factor 1000, when compared to original computations from [2].

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Posters

Heteroclinic behavior in rotating Rayleigh-Bénard convection

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We investigate numerically the appearance of heteroclinic behavior in a three-dimensional, buoyancy-driven fluid layer with stress-free top and bottom boundaries, a square horizontal periodicity with a small aspect ratio, and rotation at low to moderate rates about a vertical axis. The Prandtl number is 6.8. If the rotation is not too slow, the skewed-varicose instability leads from stationary rolls to a stationary mixed-mode solution, which in turn loses stability to a heteroclinic cycle formed by unstable roll states and connections between them. The unstable eigenvectors of these roll states are also of the skewed-varicose or mixed-mode type and in some parameter regions skewed-varicose like shearing oscillations as well as square patterns are involved in the cycle. Always present weak noise leads to irregular horizontal translations of the convection pattern and makes the dynamics chaotic, which is verified by calculating Lyapunov exponents. In the nonrotating case the primary rolls lose, depending on the aspect ratio, stability to traveling waves or a stationary square pattern. We also study the symmetries of the solutions at the intermittent fixed points in the heteroclinic cycle.

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