Exercises

Partial Differential Equations

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Problem 9

Consider the shift $(T(t)u_0)(x) := u_0(t+x)$. Is $T(t)$ a (strongly continuous) semigroup on X, if

(i) $X = L^\infty(0,\infty)$;

(ii) $X = BC^0(0,\infty)$, the bounded and continuous functions with the sup-norm;

(iii) $X = BC^1(0,\infty)$, the subset of BC^0 of functions with bounded and continuous first derivative with the sup-norm for the functions and their derivatives;

(iv) $X = BC^0_{\text{unif}}(0,\infty)$, the uniformly continuous functions with the sup-norm?

Problem 10

At points $x \in h\mathbb{Z}$ along the x-axis we consider identical mathematical pendulums moving in the yz-plane. Each pendulum is connected to its neighbours with a linear torsion spring of strength $1/h^2$. Therefore we have for the angle u at $x = nh$:

$$u''_{nh}(t) = g\sin(u_{nh}(t)) + \frac{1}{h^2}(u_{(n-1)h} - 2u_{nh} + u_{(n+1)h}).$$

Which partial differential equation can be obtained in the formal limit $h \searrow 0$?
Let w be the winding number of the pendulums around the x-axis for $-\infty < x < \infty$. Find non-constant travelling waves with $w = 0, 1, \infty$.

Problem 11

Let A be the operator $\mathrm{d}^2/\mathrm{d}x^2$ on the space $X = C^0([0,1])$ of continuous functions with domain $\mathcal{D}(A) = \{u \in C^2([0,1]) \mid u(0) = u(1) = 0\}$. Is A closed? Is the corresponding operator with Neumann boundary condition closed?
Extra. Is A on $BC^0(\mathbb{R})$ closed, without boundary conditions?

Problem 12

Let $a, b \in BC^1(\mathbb{R}, \mathbb{R})$ be positive functions. Consider a solution $u \in C^1(\mathbb{R}^2, \mathbb{R})$ of

$$a(x)\partial_x u + b(y)\partial_y u = 0.$$

Prove the existence of functions $f, g, h \in C^1(\mathbb{R}, \mathbb{R})$ such that

$$u(x, y) = f(g(x) + h(y)).$$

Tip. Make an ansatz for f, g, h and prove the identity using characteristics.