Exercises

Partial Differential Equations

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Problem 13

Let $u \in W^{1,p}([0,1])$, $1 \leq p \leq \infty$, i.e. $u, v \in L^p$ such that for all $\varphi \in C_0^\infty$

$$\int_0^1 u\varphi' = -\int_0^1 v\varphi.$$

With clever choices of ϕ, prove that $u \in C^{0,\gamma}$ for all $\gamma \leq 1 - 1/p$. (Assume u to be continuous, if necessary.)

Problem 14

Let B be the unit ball in $\mathbb{R}^n$. Consider $u_\beta : B \to \mathbb{R}$, $x \mapsto u_\beta(x) = |x|^\beta$. Find the values of β such that

(i) $u_\beta \in C^{0,\alpha}(B)$;

(ii) $u_\beta \in W^{1,p}(B)$;

Problem 15

Let $\Omega \subset \mathbb{R}^n$ be open and connected. Consider $u \in W^{1,p}(\Omega)$, $1 \leq p < \infty$, with vanishing weak derivative

$$\int_\Omega u\partial_i\eta = 0 \qquad \forall\eta \in C_0^\infty(\Omega),\ i = 1, \dots, n.$$

Prove that then u is a constant function.

Problem 16

Let $(f * g)(x) := \int_{\mathbb{R}^n} f(x-y)g(y)\,\mathrm{d}y$ the convolution and $T(t)$ the semigroup of the linear heat equation on $L^p(\mathbb{R}^n)$, $1 < p < \infty$. Prove

(i) $\|f * g\|_{L^p} \leq \|f\|_{L^1}\|g\|_{L^p}$ for $1 \leq p \leq \infty$.
Hint: Try $p = 1$ first. Then decompose $|f(x-y)| = |f(x-y)|^{1/q}|f(x-y)|^{1/p}$, for $p > 1$, and apply the Hölder inequality.

(ii) $\|T(t)\|_{L^p} \leq 1$, i.e. $T(t)$ is a contracting semigroup.