Exercises

Partial Differential Equations

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Problem 17

Motivated by the "delay equation" $\dot{x}(t) = x(t-1)$ we define recursively

$$x(t) := x(N) + \int_{N-1}^{t-1} x(s)\,\mathrm{d}s, \qquad \text{for } N < t \leq N+1,\ N = 0, 1, 2, \dots$$

with initial part $x(\cdot) = x_0(\cdot) \in X := C^0([-1,0], \mathbb{R})$. Now let

$$(T(t)x_0)(\vartheta) := x(t+\vartheta), \qquad \text{for } t \geq 0,\ -1 \leq \vartheta \leq 0.$$

Prove that $T(t)$ is a semigroup. What is the infinitesimal generator?

Problem 18

Consider the n-dimensional set $R = \{x \in \mathbb{R}^n \mid a_i \leq x_i \leq b_i,\ 1 \leq i \leq n\}$ and its $(n-1)$-dimensional projection $R' = \{x' \in \mathbb{R}^{n-1} \mid a_i \leq x'_i \leq b_i,\ 1 \leq i \leq n-1\}$. Let $1 \leq p \leq \infty$.

(i) For all $u \in C^\infty(R) \cap W^{1,p}(R)$ and all $a_n \leq \xi \leq b_n$ the inequality

$$\|u(\cdot,\xi)\|_{L^p(R')} \leq K\|u\|_{W^{1,p}(R)}$$

is satisfied with some constant $K = K(p, b_n - a_n)$.

(ii) Does this implies that the map

$$B\ :\ W^{1,p}(R) \longrightarrow L^p(R'), \quad u \longmapsto u(\cdot,\xi)$$

is a continuous, linear operator?

Problem 19

Let Ω be a bounded domain in $\mathbb{R}^n$, $\partial\Omega$ of class C^1, and $h : \partial\Omega \to \mathbb{R}$ continuous. Prove that on $u, v \in H^1(\Omega)$

$$B[u,v] := \int_\Omega (\nabla u \cdot \nabla v + kuv)\,\mathrm{d}x + \int_{\partial\Omega} huv$$

is bounded and coercive as long as $k > 0$ is chosen large enough. Let $f \in L^2$. Which partial differential equation is satisfied by a (sufficiently smooth) solution u of

$$B[u,v] = \int_\Omega f \cdot v\,\mathrm{d}x, \qquad \forall v \in H^1(\Omega).$$

In particular, discuss the boundary conditions.

Problem 20

We are interested in the limiting cases of Sobolev embeddings. Define a function

$$a_{\beta,\gamma} \, : \, (0, 1/\mathrm{e}] \to \mathbb{R}, \qquad a_{\beta,\gamma}(r) = \int_r^{1/\mathrm{e}} \rho^\beta |\log \rho|^\gamma \mathrm{d}\rho.$$

Let B be the ball in $\mathbb{R}^n$ with radius $1/\mathrm{e}$ around the origin. Consider $u_{\beta,\gamma} : B \to \mathbb{R}$, $x \mapsto u_{\beta,\gamma}(x) = a_{\beta,\gamma}(|x|)$. Find the values of β, γ such that

(i) $u_\beta \in L^q(B)$, $1 \leq q \leq \infty$;

(ii) $u_\beta \in W^{1,p}(B)$, $1 \leq p \leq \infty$;

Prove, in particular, that $W^{1,n}(B) \not\hookrightarrow L^\infty(B)$!