Exercises

Partial Differential Equations

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Problem 25

Consider the operators $A = \partial_x$ and $A = \mathrm{i}\partial_x$ with $\mathcal{D}(A) = H^1$ on the complex Hilbert space $X = L^2(\mathbb{R}, \mathbb{C})$ with scalar product

$$(u, v) := \int_{-\infty}^{+\infty} u(x)\overline{v}(x)\,\mathrm{d}x.$$

Is A

(i) symmetric?
(ii) closed?
(iii) self adjoint?

Problem 26

On $X = L^2(\mathbb{R}, \mathbb{C})$ define

$$(E(\lambda)u)(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\lambda} e^{ix\mu}\hat{u}(\mu)\,\mathrm{d}\mu$$

where $\hat{u}$, the Fourier transform of u, is given by

$$\hat{u}(\mu) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-i\mu x}u(x)\,\mathrm{d}x.$$

Prove that $E(\lambda)$ is a spectral family. Which operator is defined by $\int_{-\infty}^{+\infty} \lambda\,\mathrm{d}E(\lambda)$?

Problem 27

Let $A : \mathcal{D}(A) \subseteq X \to X$ be densely defined in the Banach space X. Prove that $\|u\|_1 := \|u\|_X + \|Au\|_X$ defines a norm on $\mathcal{D}(A)$. Prove that $\mathcal{D}(A)$ with this norm is a Banach space if, and only if, A is closed.

Problem 28

We consider the shift $(T(t)u)(x) := u(t + x)$ on $X = L^p(\mathbb{R}^+)$, $1 \leq p < \infty$. Let A be the infinitesimal generator. Prove that the resolvent set of A consists of all $\lambda \in \mathbb{C}$ with positive real part.
Hint: Discuss $(A - \lambda)u$ for $u(x) = \exp(\lambda x)$ and use the fact that the resolvent set is open.