Exercises

Partial Differential Equations

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Problem 29

We are interested in the product and chain rules for Sobolev functions on a bounded domain Ω:

(i) For $f \in H^{k,p}(\Omega)$ and $f \in H^{k,q}(\Omega)$ with $1/p + 1/q = 1$, the product $fg \in H^{k,1}(\Omega)$ and $(fg)' = f'g + g'f$ holds.

(ii) For $f \in H^{1,p}(\Omega)$ and a C^1-Diffeomorphism $\tau : \Omega' \to \Omega$ (τ, τ^{-1} are C^1-functions with bounded derivatives) prove that $f \circ \tau \in H^{1,p}(\Omega')$ and the chain rule is valid.

Problem 30

Let A be a densely defined and closed operator on the Hilbert space Y. Assume that all positive (real) λ are contained in the resolvent set of A with the estimate:

$$\|(A - \lambda)^{-1}\| \leq \lambda^{-1}, \qquad \forall \lambda \in (0, \infty).$$

Prove that A is dissipative and even maximal dissipative.
Hint: Consider the scalar product $\big((A - \lambda)u, (A - \lambda)u\big)_Y$.
Extra: Find an operator A that is not dissipative but has a resolvent set which contains all positive λ (of course without the above estimate).

Problem 31

Consider the wave equation (summation convention)

$$v_{tt} + \alpha v_t = b_{jk} v_{x_j x_k} + c_j v_{x_j} + dv, \qquad x \in \mathbb{R}^n$$

with v real, b symmetric and uniformly positive definite, α, c, d in BC^0, and b in BC^1. Which regularity of the coefficients is needed to define a semigroup for (v, v_t) in $H^{k+1} \times H^k$? Which k provides classical solutions (that means all needed derivatives are continuous) in $\mathbb{R}^3$?

Problem 32

Transform the wave equation $u_{tt} = u_{xx}$, with $x \in \mathbb{R}$, $t \geq 0$ to a symmetric hyperbolic system for $\underline{u} = (u_x, u_t, u) \in L_2(\mathbb{R})^3$. What is the infinitesimal generator? Find an explicit expression for the solution of the semigroup of the symmetric hyperbolic system.