Exercises

Partial Differential Equations

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Due Date: January 17, 2002

Problem 33

Consider strongly continuous semigroups T, S on X with infinitesimal generators A, B, respectively. Assume $T(t)S(\tau) = S(\tau)T(t)$ for all $t, \tau > 0$. Prove that $T(t)S(t)$ is a strongly continuous semigroup. Infinitesimal generator? Domain?

Problem 34

Consider the operator

$$A: \ (H^1 \times L^2)(\mathbb{R}) \supset (H^2 \times H^1)(\mathbb{R}) \to (H^1 \times L^2)(\mathbb{R}), \quad (u, v) \mapsto (v, -u_{xx}).$$

Prove that A is densely defined and closed but does not generate a strongly continuous semigroup.

Problem 35

Let H_1, H_2 be Hilbert-spaces.

$$A: \ H_1 \supset D(A) \longmapsto H_2 \qquad \text{linear, densely defined.}$$

Prove:

(i) A^* is closed.

(ii) A^* is densely defined if, and only if, A is closable. In that case, the closure of A is $\mathrm{clos}(A) = A^{**}$.

(iii) If $H = H_1 = H_2$ and A is symmetric then A is closable and $\mathrm{clos}(A)$ is symmetric.

(iv) If $H = H_1 = H_2$ and A, A^* are both symmetric then $A^* = A^{**} = \mathrm{clos}(A)$ is selfadjoint.

Problem 36

Let $\{\tau_{j,k}\}_{j,k\in\mathbb{N}}$ be a Hermitian symmetric (infinite) matrix,

$$\overline{\tau_{j,k}} \;=\; \tau_{k,j} \qquad \forall j,k \in \mathbb{N},$$

such that the rows are contained in ℓ_2:

$$\sum_{k\in\mathbb{N}} |\tau_{j,k}|^2 \;<\; \infty \qquad \forall j \in \mathbb{N}.$$

Define a linear Operator

$$T \;:\; \ell_2 \supset D(T) \longmapsto \ell_2, \qquad (Tu)_j \;:=\; \sum_{k\in\mathbb{N}} \tau_{j,k} u_k, \quad \text{for } u = \{u_k\}_{k\in\mathbb{N}}$$

with

$$D(T) \;=\; \{u \mid \{(Tu)_j\}_{j\in\mathbb{N}} \in \ell_2\}$$

Note that $(Tu)_j$ is always finite for $u \in \ell_2$. $D(T)$ is dense, it contains all finite sequences. In particular define

$$T_0 \;:\; \ell_2 \supset D(T_0) \longmapsto \ell_2, \qquad T_0 u \;:=\; Tu$$

on the dense set $D(T_0)$ of finite sequences. Prove:

(i) T_0 is symmetric.

(ii) $T_0^* \;=\; T$.

In particular, clos(T_0) is selfadjoint if, and only if, T is symmetric.

Problem 37

Find an example of a closed, symmetric, densely defined operator on a Hilbert space that is **not** selfadjoint.
Hint: Use the Operator from the previous problem and find special parameter values $\tau_{j,k}$ and special $u, v \in D(T)$ such that $(Tu, v) \neq (u, Tv)$.

Problem 38

A strongly continuous family of bounded operators $T(t)$, $t \in \mathbb{R}$, on a Banach space X is called (strongly continuous) *group* if $T(t)u$ is continuous in t, $T(0) = \mathrm{id}$, and $T(t+s) = T(t)T(s)$ for *all* real t, s (not only positive), $u \in X$.
Let A be the infinitesimal generator.

(i) Prove that A is closed and densely defined. There exists constants M, β such that

$$|(A - \lambda)^{-n}| \leq \frac{M}{(|\Re\mathrm{e}(\lambda)| - \beta)^n}$$

for all $|\Re\mathrm{e}(\lambda)| > \beta$ and $n = 1, 2, 3, \ldots$.

(ii) Prove that a given operator A generates a (strongly continuous) group $T(t)$ if (i) is satisfied.

(iii) Find an *interesting* example of a (strongly continuous) group $T(t)$.