

Exercises

**Partial Differential Equations**

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**Problem 43**

Let  $A$  be a complex  $(n \times n)$ -matrix with simple eigenvalues and  $\varphi : \mathbb{C} \rightarrow \mathbb{C}$  a holomorphic function. Define

$$\varphi(A) := \frac{1}{2\pi i} \int_{\Gamma} \varphi(\lambda)(\lambda - A)^{-1} d\lambda,$$

where  $\Gamma$  is a (positively oriented) curve around  $\text{spec}(A)$ .

- (i) Can we obtain  $\varphi(A)$  by calculating the power series of  $\varphi$  with argument  $A$ ?
- (ii) Let  $A$  be symmetric and  $E(\lambda)$  its spectral family. Does  $\phi(A) = \int_{-\infty}^{+\infty} \phi(\lambda) dE(\lambda)$  hold?
- (iii) Discuss both questions for  $A$  with non-simple eigenvalues.

**Problem 44**

Let  $A \in G(M, \beta)$  be the generator of  $T(t)$  on  $X$ . Define  $X_n := D(A^n)$  with the graph norm  $\|u\|_n := |A^n u|_X + |u|_X$  for  $n \in \mathbb{N}$ . Prove:

- (i)  $D(A^{n+1}) \subset X^n$  is dense.
- (ii) The operator  $A_n := A|_{X_n}$  with  $D(A_n) = D(A^{n+1})$  generates the restricted semigroup  $T_n(t) = T(t)|_{X_n}$  and  $A_n \in G(M, \beta)$ .

**Problem 45**

Give an example of a strongly continuous but non-analytic semigroup and its generator. Prove directly (without using the theorem) that the generator is not sectorial.

**Problem 46**

[Henry §1, ex. 8] Let  $A$  be sectorial and  $T(t)$  the generated analytic semigroup on  $X$ . Prove for every number  $m \in \mathbb{N}$ :

- (i)  $R(T(t)) \subset D(A^m)$  for all  $t > 0$ .
- (ii)  $D(A^m)$  is dense in  $X$ .