Exercises

Partial Differential Equations

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Problem 47

[Henry §1, ex. 8] Let $A \in G(M,0)$ be the generator of the strongly continuous semigroup $T(t)$ on X. Assume additionally:

$$\forall t \in (0,1] \qquad \|AT(t)\| \;\le\; M/t.$$

Prove:

(i) $\forall t \in (0,1], m \in \mathbb{N} \qquad \|(\,\mathrm{d}/\,\mathrm{d}t)^m T(t)\| \;=\; \|A^m T(t)\| \;\le\; M^m m^m t^{-m}.$

(ii) $T(t), t \ge 0$ is an analytic semigroup.

Problem 48

Let $A \in \mathcal{H}(M,\beta,\delta)$ be sectorial, $\beta < 0$, $0 < \alpha < 1$. Prove:

(i) $(-A)^{-\alpha} \;=\; \dfrac{\sin(\pi\alpha)}{\pi} \displaystyle\int_0^\infty t^{-\alpha}(t-A)^{-1}\,\mathrm{d}t$

(ii) $(-A)^{-\alpha} \;=\; \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^\infty t^{\alpha-1} T(t)\,\mathrm{d}t$

Remark: For $x \notin \mathbb{Z}$ you can use $\Gamma(x)\Gamma(1-x) = \pi/\sin(\pi x)$.

Problem 49

Let $A \in \mathcal{H}(M,\beta,\delta)$ be sectorial, $\beta < 0$, $\alpha > 0$. Prove that $(-A)^\alpha$ is a densely defined and closed operator and therefore X^α with the graph norm is a Banach space.

Problem 50

Let A be a sectorial generator of the analytic semigroup $T(t)$ and $u(t)$ be a mild solution of

$$(*) \qquad\qquad u'(t) = Au(t) + f(t)$$

on the interval $0 < t \le \tau$, $u(0) = u_0 \in X$. Assume that there exists an $\alpha > 0$ such that $f \in C^0([0,\tau], X^\alpha)$ (but no Hölder continuity!). Prove that u is a solution of $(*)$.

Problem 51

Consider the operator $Bu = u_{xx}$ on the space $X = L^2(S^1)$, i.e. with periodic boundary conditions, $S^1 = \mathbb{R}/2\pi\mathbb{Z}$. Prove that B is sectorial and describe the space X^α using conditions on the Fourier coefficients $u_k \in \mathbb{C}$ of the series

$$u(x) = \sum_{k=0}^{\infty} \Re\mathrm{e}\left(u_k \mathrm{e}^{\mathrm{i}kx}\right).$$

Problem 52

Let $A \in \mathcal{H}(M, \beta, \delta)$ be a sectorial operator on X, $\beta < 0$. Prove that

$$T(\alpha) := (-A)^{-\alpha}, \qquad \alpha \geq 0$$

is a strongly continuous (even analytic) semigroup on $\mathcal{L}(X, X)$. What is the infinitesimal generator?