# Genaral structures of Bifurcation equations: (by Abderrahim Azouani) "Seminar Symmetrische Dynamische Systeme"

Let G be a compact Lie group acting linearly on a real dimensional vector space V.
A polynomial p on $\mathbb{R}$ is called *invariant if* $p(gv) = p(v)$ *for all* $g \in G$ *and* $v \in V$.
*A polynomial map* $Q$ *on* $V$ *is called equivariant if* $Q(gv) = gQ(v)$.
Theorem 1: *The set* $R_G$ *of G-invariant polynomials is finitely generated as* $\mathbb{R}$-*algebra.*
Lemma 1: *Let* $p_1, .., p_r$ *be G-invariant polynomials then:*
$R_G \cap \sum_{i=1}^{r} P(V)p_i = \sum_{i=1}^{r} R_G p_i$.
Lemma 2: *If* $A_+$ *is finitely generated as A-module then A is finitely generated as* $\mathbb{R}$-*algebra.*
Definition 1 : *A commutative ring R is called Noetherian if the following equivalent conditions are satisfied:*
*1. Any ideal I of R is of finite type.*
*2. If* $I_1 \subset I_2 ... \subset I_n ...$ *is an ascending sequence of ideals of R then there is N such that* $I_n = I_{n+1}$ *for* $n > N$.
Theorem 2 *(Hilbert): The polynomial* $R[x]$ *is also Noetherian.*
*The theorem implies that every G-invariant polynomial can be written as a polynomial in the generators of the ring.*
Lemma 3: *The set of equivariant polynomial maps forms a module M over the ring* $R_G$. *As an* $\mathbb{R}$-*Algebra, it is finitely generated.*
*Let* $\pi_1, ..., \pi_s$ *be a set of generators of the algebra* $R_G$ *which is called Hilbert basis. It is not uniquely defined but there is a minimal number of genrators for a representation* $\rho$ *of G denoted by* $m(\rho)$.
*Let* $e_1, ...e_d$ *be a set of generators for M over* $R_G$.

$$\begin{tikzcd}
\mathbb{R}^n \arrow[rr, "H"] \arrow[dr, "o"'] & & H(\mathbb{R}^n) \subset \mathbb{R}^{m(\rho)} \\
& \mathbb{R}^n/G \arrow[ur, "H'"'] &
\end{tikzcd}$$

Figure 1: $o$ is the canonical mapping and $H'$ is a homeomorphism

*Theorem 3: Any equivariant map $Q$ has the form:*
$Q(x) = \sum_{i=1}^{d} f_i(\pi_1, ...\pi_s)\; e_i$.

*Definition 2 : A Hilbert map is defined from $V$ to $\mathrm{I\!R}^{m(\rho)}$:*
$H(x) = (\pi_1, ..., \pi_{m(\rho)})$

*Theorem 4: A Hilbert map has the following properties:*
*(i) $H$ is proper, (ii) $H$ separates orbits and (iii) the digramm of (figure 1) commutes.*

**The results for polynomial maps can be extended to the smooth case in a natural way:**

**Theorem 5: Let $R_G^\infty$ denote the set of all smooth invariant functions.**
**If $g \in R_G^\infty$ there exists $\hat{g} : \mathrm{I\!R} \to \mathrm{I\!R}^s$ such that: $g(x) = \hat{g}(\pi_1(x), ..., \pi_s(x))$.**

**Theorem 6 (Schwarz) : The map $H^*$ which is defined from $C^\infty(\mathrm{I\!R}^{j(\rho)})$ to $C_G^\infty(V, \mathrm{I\!R})$ is surjective.**

**Let $M_G(V)$ the space of smooth equivariant maps on V or more generally:**
**$M_G^\infty(V_1, V_2) := \{f : V_1 \to V_2 | f$ is equivariant w.r.t. $\rho_1, \rho_2\}$, where $\rho_{1,2}$ are representations on $V_{1,2}$.**

**Theorem 7 (Poénaru) :**
**(i) $M_G(V_1, V_2)$ is a module over $R_G(V_1)$.**
**(ii) $M_G^\infty(V_1, V_2)$ is a module over $R_G^\infty(V_1)$.**
**(iii) $M_G(V_1, V_2)$ is finitely generated as $R_G(V_1)$-module.**
**(iv) $M_G^\infty(V_1, V_2)$ is finitely generated as $R_G^\infty(V_1)$-module.**