The Equivariant Hopf Theorem

Faouzi LAKRAD

Consider the following system of ordinary differential equations

$$\frac{dx}{dt} = f(x, \lambda), \tag{1}$$

where $f : \mathbb{R}^{2m} \times \mathbb{R} \to \mathbb{R}^{2m}$ and $f(0,0) = 0$.

- **Definition.** The vector space W is Γ-simple if either:
(a) $W = V \oplus V$ where V is an absolutely irreducible representation of Γ, or
(b) Γ acts irreducibly but not absolutely irreducibly on W.

- **Equivariant Hopf Theorem** [1]
Let a compact Lie group Γ act Γ-simply on $\mathbb{R}^{2m}$. Assume that
(a) $f : \mathbb{R}^{2m} \times \mathbb{R} \to \mathbb{R}^{2m}$ is Γ-equivariant. Then $f(0, \lambda) = 0$ and $(df)_{0,\lambda}$ has eigenvalues $\sigma(\lambda) \pm i\rho(\lambda)$ each of multiplicity m.
(b) $\sigma(0) = 0$ and $\rho(0) = 1$.
(c) $\sigma'(0) \neq 0$ (the eignevalue crossing condition).
(d) $\Sigma \subseteq \Gamma \times S^1$ is an isotropy subgroup and dim $Fix(\Sigma) = 2$.
Then there exists a unique branch of periodic solutions to (1) with period near 2π having Σ as their group of symmetries.

- **Idea of the Proof.** The main idea is to use Liapunov-Schmidt Reduction to prove the existence of periodic solutions. To do this, a mapping, whose zeros correspond to periodic solutions of the original system of differential equations (1), is set up.

References

[1] M. Golubitsky, I. Stewart and D. G. Schaeffer, *Singularities and groups in bifurcation theory*. Volume II. Applied Mathematical Sciences, 69. New York: Springer-Verlag, 1988.