hyperbolic areas A_j over $[a_j, a_{j+1}]$ and B_j over $[b_j, b_{j+1}]$, it is straightforward to calculate the corresponding inequalities:

$$(a_{j+1} - a_j)\frac{1}{a_{j+1}} < A_j < (a_{j+1} - a_j)\frac{1}{a_j} \quad \text{and} \quad (b_{j+1} - b_j)\frac{1}{b_{j+1}} < B_j < (b_{j+1} - b_j)\frac{1}{b_j}$$

Substituting the values $b_j = va_j$ into the second set of inequalities gives

$$(a_{j+1} - a_j)\frac{1}{a_{j+1}} < B_j < (a_{j+1} - a_j)\frac{1}{a_j}.$$

Thus both hyperbolic regions are squeezed between rectangles of the same areas. Because both intervals can be divided into subintervals as small as desired, it follows that the two hyperbolic areas are equal.

When the Belgian Jesuit Alfonso Antonio de Sarasa (1618–1667) read Gregory's work in 1649, he immediately noticed that this calculation implied that the area $A(x)$ under the hyperbola from 1 to x had the logarithmic property $A(\alpha\beta) = A(\alpha) + A(\beta)$. (Because the ratio $\beta : 1$ equals the ratio $\alpha\beta : \alpha$, the area from 1 to β equals the area from α to $\alpha\beta$. Because the area from 1 to $\alpha\beta$ is the sum of the areas from 1 to α and from α to $\alpha\beta$, the logarithmic property is immediate.) Thus if one could calculate the area under the hyperbola $xy = 1$, one could calculate logarithms. The search for means of calculating these areas led to the power series methods of Newton and others in the 1660s, methods which were instrumental in Newton's version of the calculus.

12.3 POWER SERIES

In 1668, Nicolaus Mercator (1620–1687) published his *Logarithmotechnica* (*Logarithmic Teachings*), in which appeared the power series expansion for the logarithm. Mercator, having read the hint of de Sarasa that the logarithm was related to the area under a hyperbola and having learned from Wallis how to calculate certain ratios of infinite sums of powers, decided to calculate $\log(1 + x)$ (the area A under the hyperbola $y = 1/(1 + x)$ from 0 to x) by using such infinite sums. He divided the interval $[0, x]$ into n subintervals of length x/n and approximated A by the sum

$$\frac{x}{n} + \frac{x}{n}\left(\frac{1}{1 + \frac{x}{n}}\right) + \frac{x}{n}\left(\frac{1}{1 + \frac{2x}{n}}\right) + \cdots + \frac{x}{n}\left(\frac{1}{1 + \frac{(n-1)x}{n}}\right).$$

Since each term $\frac{1}{1 + (kx/n)}$ is the sum of the geometric series $\sum_{j=0}^{\infty}(-1)^j\left(\frac{kx}{n}\right)^j$, it follows that

$$A \approx \frac{x}{n} + \frac{x}{n}\sum_{j=0}^{\infty}(-1)^j\left(\frac{x}{n}\right)^j + \frac{x}{n}\sum_{j=0}^{\infty}(-1)^j\left(\frac{2x}{n}\right)^j + \cdots + \frac{x}{n}\sum_{j=0}^{\infty}(-1)^j\left(\frac{(n-1)x}{n}\right)^j$$

$$= n\frac{x}{n} - \frac{x^2}{n^2}\sum_{i=1}^{n-1} i + \frac{x^3}{n^3}\sum_{i=1}^{n-1} i^2 + \cdots + (-1)^j\frac{x^{j+1}}{n^{j+1}}\sum_{i=1}^{n-1} i^j + \cdots$$

$$= x - \frac{\sum_{i=1}^{n-1} i}{n \cdot n}x^2 + \frac{\sum_{i=1}^{n-1} i^2}{n \cdot n^2}x^3 + \cdots + (-1)^j\frac{\sum_{i=1}^{n-1} i^j}{n \cdot n^j}x^{j+1} + \cdots.$$

By Wallis' results, the coefficient of x^{k+1} in this expression is equal to $1/(k+1)$ if n is infinite. Therefore,

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots,$$

a power series in x which enabled actual values of the logarithm to be calculated easily.

Power series for other transcendental functions were discovered by James Gregory (1638–1675) in Scotland around 1670 and communicated to John Collins (1625–1683), the secretary of the Royal Society, without any indication of how they were discovered. For example, in a letter of December 19, 1670, Gregory wrote that the arc whose sine is B (where the radius of the circle is R) is expressible as

$$B + \frac{B^3}{6R^2} + \frac{3B^5}{40R^4} + \frac{5B^7}{112R^6} + \frac{35B^9}{1152R^8} + \cdots.$$ [20]

In modern terminology, Gregory's series is the series for $1/R \arcsin B/R$, which, if $R = 1$, can be written

$$\arcsin x = x + \frac{x^3}{6} + \frac{3x^5}{40} + \frac{5x^7}{112} + \frac{35x^9}{1152} + \cdots.$$

Similarly, in a letter of February 15, 1671, Gregory included, among others, the series for the arc y given the tangent x and vice versa, written in modern notation as:

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots$$

$$\tan y = y + \frac{y^3}{3} + \frac{2y^5}{15} + \frac{17y^7}{315} + \frac{3233y^9}{181440} + \cdots.$$ [21]

However Gregory derived these series, it turns out that the arctangent series, as well as series for the sine and cosine which Newton found in the mid-1660s, had been discovered in southern India perhaps 200 years earlier. These series appear in Sanskrit verse in the *Tantrasangraha-vyākhyā* (c. 1530), a commentary on a work by Kerala Gargya Nīlakaṇṭha (1445–1545) of some 30 years earlier. Unlike the situation for many results of Indian mathematics, a detailed proof of these results exists, in the *Yuktibhāṣa*, a work in Malayalam, the language of Kerala, the southwestern region of India. The *Yuktibhāṣa*, written by Jyesthadeva (1500–1610), credits the arctangent series to the earlier mathematician Madhava (1340–1425), who lived near Cochin.

The Sanskrit verse giving the arctangent series may be translated as follows:

> The product of the given sine and the radius, divided by the cosine, is the first result. From the first [and the second, third, etc.] results, obtain [successively] a sequence of results by taking repeatedly the square of the sine as the multiplier and the square of the cosine as the divisor. Divide the above results in order by the odd numbers one, three, etc. [to get the full sequence of terms]. From the sum of the odd terms subtract the sum of the even terms. The result becomes the arc. In this connection . . . the sine of the arc or that of its complement, whichever is smaller, should be taken here [as the given sine]; otherwise the terms obtained by the [above] repeated process will not tend to the vanishing magnitude.[22]

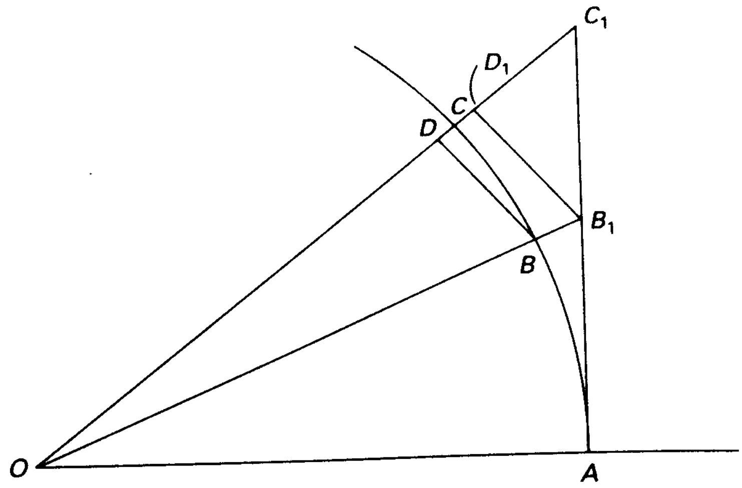

FIGURE 12.14
Jyesthadeva's derivation of the arctangent series.

It is not difficult to translate these words into the modern symbolism of the same arctangent series which Gregory found, noting that the author has realized that convergence only occurs when $\tan\theta \le 1$.

Jyesthadeva's proof of the validity of the arctangent series begins with the following lemma where, for simplicity, the radius of the circle is set equal to 1:

Lemma. *Let BC be a small arc of a circle with center O. If OB, OC meet the tangent at any point A of the circle in the points B_1, C_1 respectively, then arc BC is given approximately by arc $BC \approx B_1C_1/(1 + AB_1^2)$ (Figure 12.14).*

If perpendiculars BD, B_1D_1 are drawn to OC, it follows by similarity that $\frac{BD}{B_1D_1} = \frac{OB}{OB_1} = \frac{1}{OB_1}$ and $\frac{B_1D_1}{B_1C_1} = \frac{OA}{OC_1} = \frac{1}{OC_1}$ and therefore that $BD = \frac{B_1C_1}{OB_1 \cdot OC_1}$. When arc BC is very small, $OB_1 \approx OC_1$ and therefore arc $BC \approx BD = \frac{B_1C_1}{OB_1^2} = \frac{B_1C_1}{1 + AB_1^2}$.

Dividing the tangent $t = AC_1$ to arc AC into n equal parts, applying the lemma to each in turn, and then letting n get indefinitely large gives

$$
\begin{aligned}
\arctan t &= \lim_{n\to\infty} \sum_{r=0}^{n-1} \frac{t/n}{1 + \left(\frac{rt}{n}\right)^2} \\
&= \lim_{n\to\infty} \sum_{r=0}^{n-1} \frac{t}{n}\left[1 - \left(\frac{rt}{n}\right)^2 + \left(\frac{rt}{n}\right)^4 - \cdots + (-1)^k\left(\frac{rt}{n}\right)^{2k} + \cdots\right] \\
&= \lim_{n\to\infty} \left[\frac{t}{n} + \frac{t}{n}\left(1 - \frac{t^2}{n^2} + \frac{t^4}{n^4} - \cdots\right) + \frac{t}{n}\left(1 - \frac{2^2t^2}{n^2} + \frac{2^4t^4}{n^4} - \cdots\right)\right. \\
&\qquad \left. + \frac{t}{n}\left(1 - \frac{3^2t^2}{n^2} + \frac{3^4t^4}{n^4} - \cdots\right) + \cdots + \frac{t}{n}\left(1 - \frac{(n-1)^2t^2}{n^2} + \frac{(n-1)^4t^4}{n^4} - \cdots\right)\right] \\
&= \lim_{n\to\infty} \left[t - \frac{t^3}{n^3}(1^2 + 2^2 + \cdots + (n-1)^2) + \frac{t^5}{n^5}(1^4 + 2^4 + \cdots + (n-1)^4) - \cdots\right]
\end{aligned}
$$

To complete the derivation, Jyesthadeva needed to deal with sums of integral powers. Like ibn al-Haytham, he showed that

$$n\sum_{j=1}^{n} j^{p-1} = \sum_{j=1}^{n} j^p + \sum_{k=1}^{n-1}\sum_{j=1}^{k} j^{p-1},$$

a result which implied Wallis' theorem that

$$\lim_{n\to\infty} \frac{1}{n^{p+1}}\sum_{j=1}^{n} j^p = \frac{1}{p+1}.$$

Substituting that into the earlier formula gave him the final result that

$$\arctan t = t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \cdots.$$

Why were Hindu authors interested in this series? Their main goal seems to have been the calculation of lengths of circular arcs, values of which were necessary for astronomical purposes. This series permitted that calculation. For example, direct substitution of $t = 1$ gives $\pi/4 = 1 - 1/3 + 1/5 - 1/7 + \cdots$. Because this series converges very slowly, however, it was necessary to make various modifications. Thus, the *Tantrasaṅgraha-vyākhyā* contains other series whose convergence is considerably more rapid, including

$$\frac{\pi}{4} = \frac{3}{4} + \frac{1}{3^3 - 3} - \frac{1}{5^3 - 5} + \frac{1}{7^3 - 7} - \cdots.$$

Interestingly enough, it was the same question of determining arc length of a curve which brought European authors to the realization that the tangent problem and the area problem were related.

.4 RECTIFICATION OF CURVES AND THE FUNDAMENTAL THEOREM

Descartes stated in his *Geometry* that the human mind could discover no rigorous and exact method of determining the ratio between curved and straight lines, that is, of determining exactly the length of a curve. Only two decades after Descartes wrote those words, however, several human minds proved him wrong. Probably the first rectification of a curve was that of the semicubical parabola $y^2 = x^3$ by the Englishman William Neile (1637–1670) in 1657 acting on a suggestion of Wallis. This was followed within the next two years by the rectification of the cycloid by Christopher Wren (1632–1723), the architect of St. Paul's Cathedral and much else in London, and the reduction of the rectification of the parabola to finding the area under a hyperbola by Huygens. The most general procedure, however, was that by Hendrick van Heuraet (1634–1660(?)), which appeared in van Schooten's 1659 Latin edition of Descartes' *Geometry*.

.4.1 The Work of van Heuraet

Van Heuraet began his paper ***De transmutatione curvarum linearum in rectas*** (***On the transformation of curves into straight lines***) by showing that the problem of constructing