projects the cone with its cutting plane onto the yz plane. The line segment fg representing the diameter of the ellipse is divided into 12 equal parts, and both vertical and horizontal lines are drawn through the division points. At each of the 11 points i, the horizontal line represents part of the diameter of the circular section C_i made by a horizontal cutting plane. The two points of intersection of this circle with the ellipse are symmetrically located on the ellipse with respect to its diameter and therefore determine the width w_i of the ellipse there. The projection of the cone onto the xy plane then consists of this series of concentric circles C_i. The continuation of each vertical line becomes a chord in the corresponding circle whose length is w_i. Dürer thus has a rough projection of the ellipse. The outline of the ellipse is, however, not symmetric about its minor axis, since this projection is not taken from a direction perpendicular to the plane of the ellipse itself. But when Dürer attempts to draw the ellipse from its projection, he simply transfers the line segment representing the axis of the ellipse to a new vertical line fg, divides it at the same points i, draws horizontal line segments through each of width w_i, and then sketches the curve through the ends of these line segments. Dürer's drawing is therefore in error, because the curve is wider at the bottom than at the top. A possible reason Dürer did not realize that the ellipse should be symmetric about its minor axis is that the centerline of the cone, around the projection of which all the circles are drawn, does not pass through the center of the ellipse. Although one can prove that $w_i = w_{12-i}$ $(i = 1, 2, 3, 4, 5)$ by an analytic argument, Dürer probably believed that the ellipse was in fact egg-shaped—he does call it an Eierlinie—because the cone itself widens toward the bottom.[13]

After describing the construction and representation by projections of other space curves, Dürer continued in the second book of the *Underweysung* to describe methods for constructing various regular polygons, both exact ones using the classical tools of straightedge and compass and approximate ones, taken from the tradition of artisans. Thus the work, which was published in Latin several years after its German edition, served both to introduce the artisans to the Greek classics and also to familiarize professional mathematicians with the practical geometry of the workshop. The third book of the text was purely practical, showing how geometry could be applied in such varied fields as architecture and typography. Here Dürer suggested new types of columns and roofs as well as the methods of accurately constructing both Roman and Gothic letters. In the final book Dürer returned to more classical problems and dealt with the geometry of three-dimensional bodies. In particular, he presented a construction of the five regular solids by paper folding, a method still found in texts today, as well as similar procedures for certain semiregular solids. He also presented other problems of construction, including that of doubling the cube, before concluding the work with the basic rules for the perspective drawing of these solid figures.

10.2 GEOGRAPHY AND NAVIGATION

Two related aspects of mathematics discussed by Dee and extremely important to the world of the sixteenth century were geography and navigation. "The art of Navigation demonstrates how, by the shortest good way, by the aptest direction, and in the shortest time, a sufficient ship, between any two places (in passage navigable) assigned, may be conducted; and in all storms and natural disturbances chancing, how to use the best possible means, whereby to recover the place first assigned."[14] In the fifteenth and sixteenth centuries, Europeans were exploring the rest of the world, and methods of navigation were of central importance. The country that could employ new techniques well had great advantages in the quest for new colonies and their attendant natural resources.

The major problem of navigation on the seas was the determination of the ship's latitude and longitude at any given time. The first of these was not too difficult. One's latitude, in the northern hemisphere, was equal to the altitude of the north celestial pole, and this was marked, approximately, by Polaris, the pole star. A good approximation of the latitude was found simply by taking the altitude of that star. An alternate method of finding latitude was by observation of the sun. The zenith distance of the sun at local noon is equal to the latitude minus the sun's declination. Navigators of the fifteenth century had accurate tables of the declination for any day of the year, so they needed only to take a reading of the sun's altitude at noon. This altitude was, of course, the highest altitude of the day and could be determined by finding the shortest shadow of a standard pole.

The determination of longitude was much more difficult. Knowing the difference between the longitudes of two places is equivalent to knowing the difference between their local times, because 15° of longitude is equivalent to one hour. Theoretically, if one had a clock set to the time at a place of known longitude and could determine when, on that clock, local noon occurred at one's current location, the difference in time would enable one to make a determination of longitude. Alternatively, one could compare the known time of an astronomical event, such as an eclipse of the moon, at the place of known longitude with its local time at one's current location. Unfortunately, these methods could not work given the state of time keeping devices in the Renaissance. They were simply not accurate enough, especially if operated on the moving decks of a ship at sea. When Columbus attempted to determine longitude on his second voyage to America in 1494 using an eclipse of the moon, his error was about 18°. Despite rewards offered by various European governments for the invention of accurate methods of determining longitude at sea, the problem remained unsolved until the eighteenth century, when an accurate marine clock was finally devised.

Given these difficulties of finding one's location at sea, it is not surprising that seamen often used methods of "guesstimation" rather than mathematical astronomy. While scholars were aware that a great circle route was the shortest distance between two points, sailors generally preferred to sail to the latitude of their destination as quickly as possible and then head due east or west until they reached land. Whatever the method of navigating, however, the seamen needed accurate maps. Dee called the making of these maps Geography: "Geography teaches ways by which in sundry forms (as spherical, plane, or other) the situation of cities, towns, villages, forts, castles, mountains, woods, havens, rivers, creeks, and such other things, upon the outface of the earthly globe . . . may be described and designed in commensurations analogical to nature and verity, and most aptly to our view, may be represented."[15]

Maps had been drawn since antiquity. Ptolemy in his *Geography* had analyzed some of the problems of mapping the round earth onto flat paper, had exhibited the longitude and latitude of the known localities of the inhabited world, and had included some 26 regional maps as well as a world map. To construct his maps, he needed to use some form of projection, that is, some way of constructing a function from a portion of the earth's spherical surface to a flat piece of paper. Presumably, Ptolemy wanted the resultant maps to represent the shape of the land masses depicted as closely as possible. In any case, a projection is determined by the grid of longitude and latitude lines, generally known as meridians and parallels, respectively. For his regional maps, Ptolemy simply used a rectangular grid for these lines. On the spherical earth, however, because the spacing of the meridians depends on the latitude, he chose a scale in the two directions so that it corresponded approximately to the ratio of the length of one degree of longitude on the middle parallel of the map to one degree of latitude. This ratio is equal to $MN : AB$ (because the length of a degree of longitude at the equator is equal to that of a degree of latitude), which in turn equals $NP : BC$, $NP : NC$, and finally $\cos \phi$ (Figure 10.8). For example, because Ptolemy's map of Europe reaches from latitude 42° to latitude 54°, the given ratio should be approximately $\cos 48° = .6691$, or 2 : 3.

For his world map, which included only what he calculated to be 180° of longitude, stretching from the Strait of Gibralter to China, Ptolemy chose two different methods. In the first, the parallels were represented by concentric circles centered on the north pole while the meridians were straight lines from the pole, which was not included on the map. Ptolemy realized that this projection could not preserve the proper ratio between degrees of longitude and degrees of latitude, except within a small region of a particular parallel, which he took as the parallel of Rhodes, 36°. Thus, as in all projections of a major portion of the earth's surface, some distortion was inevitable. Ptolemy later developed a more natural appearing map by modifying the meridians into circular arcs as well. This map gives a correct representation of distance on three selected parallels through which the circular arcs are drawn, but still has distortion far from the center of the map (Figure 10.9).

Because it is impossible to make an absolutely correct map on a flat piece of paper, the mapmaker always needs to make some choice of the particular qualities of the projection desired. The mapmaker can choose to preserve areas or shapes or directions or distances. The larger the portion of the earth's surface to be represented, the more difficult it is for the map to have several of these qualities, even approximately. In general, the maps used by seamen during the early Renaissance were constructed by using a different criterion, ease of drawing. These "plane charts" used a rectangular grid for parallels and meridians, with the same scale on each. Because the distances between the meridians were the same at all latitudes, and because the true distance depends on the cosine of the latitude, shapes on these maps had the appearance of being elongated in the horizontal direction. Thus shape was not preserved and more important for the sailor, lines of con-

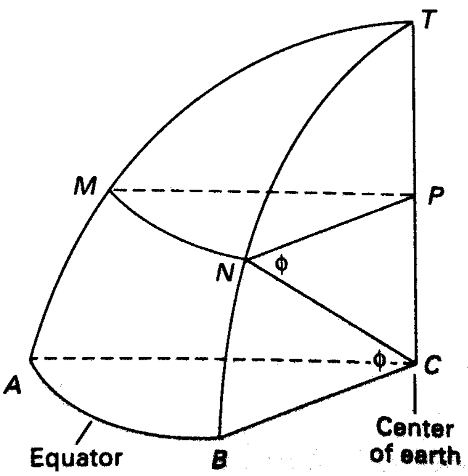

FIGURE 10.8
Relation of length of a degree of longitude at latitude ϕ to that of a degree of longitude at

FIGURE 10.9
Ptolemy's world map for the 1552 Basel edition of his *Geography*. (Source: Smithsonian Institution Photo No. 90-15779)

stant compass bearing, called rhumbs, were not represented by straight lines. When such maps were of relatively small areas, the rhumb lines were straight enough and were often drawn in for each of 8 or 16 compass directions. But as long sea voyages became increasingly common, improvements were required.

One of the first to attempt to apply mathematics to the improvement of mapmaking methods was Pedro Nuñez, in his *Tratado da sphera* of 1537. He discovered that on a sphere a rhumb line or **loxodrome**, as it is now called, becomes a spiral terminating at the pole. Using globes for navigation, however, was inconvenient because they could not be made large enough. Nuñez therefore attempted to develop a map in which loxodromes were straight lines. For accuracy, however, it was necessary that the meridians converge near the poles. Although Nuñez was able to design a device that enabled sailors to measure the number of miles in a degree along each parallel, he was not able to solve the problem he had set.

FIGURE 10.10
Mercator on a Belgian stamp.

By 1569, Nuñez's problem was solved from a slightly different point of view by Gerard Mercator (1512–1594), with a new projection known ever since as Mercator's projection (Figure 10.10). Both parallels and meridians were represented by straight lines on this map. To compensate for the "incorrect" spacing of the meridians, therefore, Mercator increased the spacing of the parallels toward the poles. He claimed that on his new map rhumb lines were now straight and a navigator could simply lay a straightedge on his map between his origin and his destination to determine the constant compass bearing to follow. Mercator did not explain the mathematical principle he followed for increasing the distance between the parallels, and some believe that he did it by guesswork alone. Not

FIGURE 10.11
A world map in Mercator projection on a Canadian stamp.

until the work of Edward Wright (1561–1615), *On Certain Errors in Navigation* (1599), did an explanation of Mercator's methods appear in print (Figure 10.11).

Recall that the ratio of the length of a degree of longitude at latitude ϕ to one at the equator is equal to $\cos \phi$. If meridians are straight lines, then, the distances between them at a latitude ϕ are stretched by a factor of $\sec \phi$. For loxodromes to be straight on such a map, then, the vertical distances must also be stretched by the same factor. Because $\sec \phi$ varies at each point along a meridian, the stretching factor needs to be considered for each small change of latitude. If we denote by $D(\phi)$ the distance on the map between the equator and the parallel of latitude ϕ, the change dD in $D(\phi)$ caused by a small change $d\phi$ in ϕ is determined by $dD = \sec \phi d\phi$. Because the same factor applies horizontally as well, any "small" region on the globe will be represented on the map by a "small" region of the same shape. The angle at which a line crosses a meridian on the globe will be transformed into that same angle on the map and loxodromes will be straight. It follows from this argument that, in modern terminology, the map distance between the equator and the parallel at latitude ϕ is given by

$$D(\phi) = \int_0^{\phi} \sec \phi d\phi, \qquad (10.1)$$

where the radius of the globe is taken as 1. Wright, of course, did not use integrals. He took for his $d\phi$ an angle of $1'$ and computed a table of what he called "meridional parts" by adding the products $\sec \phi d\phi$ for latitudes up to 75°. $D(\phi)$ can be calculated by calculus techniques as

$$\ln(\sec \phi + \tan \phi) \qquad \text{or} \qquad \ln\left(\tan\left(\frac{\phi}{2} + \frac{\pi}{4}\right)\right).$$[16]

John Dee met Mercator on one of his trips to the continent. He returned with several of Mercator's globes and probably conferred with Wright concerning the mathematical details of Mercator's projection. Thus he was involved in the process of making maps "analogical to nature." Mercator's map, although well suited for navigation, was unfortunately not "analogical to nature" for regions far from the equator. The spacing out of the parallels greatly increased the relative size of such regions. The popularity of the map led generations of students to believe, for example, that Greenland is larger than South America. Nevertheless, its simplicity of use made it the prime sea chart during the age of European exploration.

10.3 ASTRONOMY AND TRIGONOMETRY

According to Dee, "Astronomy is an art mathematical which demonstrates the distance, magnitudes, and all natural motions, appearances, and passions proper to the planets and fixed stars, for any time past, present and to come, in respect of a certain horizon, or without respect of any horizon. By this art we are certified of the distance of the starry sky and of each planet from the center of the earth, and of the greatness of any fixed star seen, or planet, in respect of the earth's greatness."[17] Thus, the purpose of astronomy is to predict the motions of the heavenly bodies as well as to determine their sizes and distances. A related art is Cosmography, "the whole and perfect description of the heav-