Homework assignment

Dynamical Systems

Bernold Fiedler, Stefan Liebscher
http://dynamics.mi.fu-berlin.de/lectures/

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Problem 1: Consider a flow $\Phi(t,x) = \Phi_t(x)$ on the real axis, $x \in \mathbb{R}$.

(i) Prove or disprove: periodic orbits do not exist (except equilibria), i.e.

$$\forall\, x_0 \in \mathbb{R} \;\Big(\, \exists\, p > 0 \;:\; \Phi(p, x_0) = x_0 \quad \Longrightarrow \quad \forall\, t \in \mathbb{R} \;:\; \Phi(t, x_0) = x_0 \,\Big)$$

(ii) What are possible α-limits and ω-limits of an orbit of Φ? (Consider bounded and unbounded orbits.)

Problem 2: Consider a flow Φ on $\mathbb{R}^n$, $n \geq 2$. The orbit $\Phi_t(x_0)$ of x_0 is assumed to have arbitrarily small periods, i.e.

$$\forall\, \varepsilon > 0 \;\; \exists\, 0 < p < \varepsilon \;:\; \Phi_p(x_0) = x_0.$$

Prove: x_0 is an equilibrium.

Problem 3: Consider

$$\Phi_t(x) \;:=\; \begin{cases} \dfrac{1}{\frac{1}{x} - t} & 0 \neq x \neq 1/t \\[2ex] 0 & 0 = x \end{cases}$$

for $x \in \mathbb{C}$ and $t \in \mathbb{R}$.

(i) Check the local flow properties for Φ_t and determine the minimal time $\underline{t}(x)$ and the maximal time $\overline{t}(x)$ of existence for every $x \in \mathbb{C}$.

(ii) Which vectorfield is associated to Φ_t?

(iii) Find all bounded / unbounded / stationary / periodic / homoclinic / heteroclinic trajectories.

(iv) Determine the limit sets $\alpha(x)$ and $\omega(x)$ for every $x \in \mathbb{C}$.

Problem 4: Prove that linear flows commute if, and only if, their linear vector fields commute. In particular consider $(N \times N)$-matrices A, B and flows

$$\Phi_t \;:=\; \mathrm{e}^{At}, \qquad \Psi_t \;:=\; \mathrm{e}^{Bt} \;:=\; \sum_{k=0}^{\infty} \frac{1}{k!} B^k t^k.$$

and prove that

$$AB \;=\; BA \qquad \Longleftrightarrow \qquad \forall t \in \mathbb{R} \;:\; \Phi_t \Psi_t = \Psi_t \Phi_t.$$

Hint: Consider $\left.\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right|_{t=0} (\Phi_t \Psi_t - \Psi_t \Phi_t)$.