Homework assignment

Dynamical Systems

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http://dynamics.mi.fu-berlin.de/lectures/

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Problem 5: Consider the vector field $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$\dot{x} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} x,$$

with $a, b \in \mathbb{R}$. Transform this linear differential equation into polar coordinates:

$$x = \begin{pmatrix} r\cos\phi \\ r\sin\phi \end{pmatrix},$$

with $r > 0$, $\phi \in \mathbb{R}/2\pi\mathbb{Z}$. Choose $b \neq 0$ arbitrarily and sketch phase portraits in (r, ϕ)-coordinates and in x-coordinates for $a < 0$, $a = 0$, $a > 0$.

Problem 6: Let $\Phi_{s,t} : \mathbb{R}^N \to \mathbb{R}^N$ be a *periodic* evolution with period $p > 0$, i.e.

$$\text{for all } t, s \in \mathbb{R} : \quad \Phi_{t+p,s+p} = \Phi_{t,s}.$$

Consider the *stroboscope* map $\Pi : \mathbb{R}^N \to \mathbb{R}^N$,

$$\Pi(x) = \Phi_{p,0}(x).$$

Prove:

(i) for all $k \in \mathbb{N} : \quad \Phi_{kp,0} = \Pi^k$;

(ii) for each $t \in \mathbb{R}$ there exists a change of coordinates $\Psi : \mathbb{R}^N \to \mathbb{R}^N$ such that for all $k \in \mathbb{N} : \quad \Phi_{t+kp,t} = \Psi^{-1}\Pi^k\Psi$. Determine Ψ.

Problem 7: Consider the initial-value problem

$$\dot{x}(t) = x(t)^2, \qquad x(0) = x_0 = 1.$$

We know from class that the solution blows up in finite time. Discuss the discretizations

(i) explicit Euler: $x_{n+1} = x_n + \varepsilon x_n^2$,

(ii) implicit Euler: $x_{n+1} = x_n + \varepsilon x_{n+1}^2$.

In particular, calculate numerical solutions for several (small) $\varepsilon > 0$ and compare with the exact solution. Do the discretizations explode at finite time? Why? How? Explain!

Extra credit: On which time interval do the discretized solutions converge to the exact solution for $\varepsilon \searrow 0$?

Problem 8: Consider the system

$$\vartheta'(t) \;=\; (\vartheta_{\text{ext}} - \vartheta(t)) + \sigma(\vartheta(t))\,(\vartheta_{\text{heat}} - \vartheta(t))\,,$$

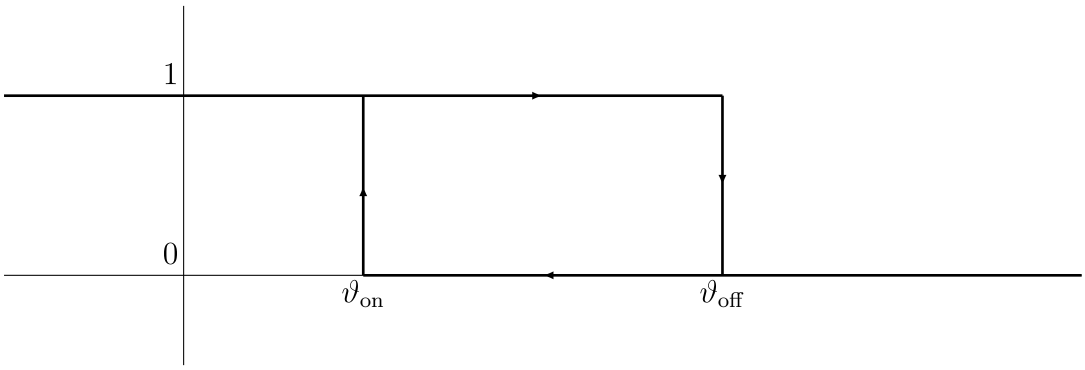
modelling the heating system of a room.

Here $\vartheta(t) \in \mathbb{R}$ is the room temperature, the constants $\vartheta_{\text{ext}}, \vartheta_{\text{heat}} \in \mathbb{R}$, $\vartheta_{\text{ext}} < \vartheta_{\text{heat}}$, represent the external temperature and the temperature of the heater. Control of the heater is modeled by the operator $\sigma : \mathbb{R} \to \{0, 1\}$.

(i) Determine the equilibria of the system if the heater is always switched off, $\sigma(\vartheta) \equiv 0$. Sketch the phase portrait.

(ii) Determine the equilibria of the system if the heater is always switched on, $\sigma(\vartheta) \equiv 1$. Sketch the phase portrait.

(iii) Discuss the system, if the heater is switched on, whenever the temperature drops below $\vartheta_{\text{on}} = 10$, and switched off, whenever the temerature rises above $\vartheta_{\text{off}} = 30$, i.e. σ is a hysteresis operator:



Find parameter regions for $\vartheta_{\text{ext}}, \vartheta_{\text{heat}}$ with different behavior of the system.