Homework assignment

Dynamical Systems

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http://dynamics.mi.fu-berlin.de/lectures/

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Problem 29: Let $f : X \to X = \mathbb{R}^n$ be locally Lipschitz continuous and $J(x_0) = (t_-(x_0), t_+(x_0))$ the maximal interval of existence of the solution to the initial-value problem

$$\dot{x}(t) = f(x(t)), \qquad x(0) = x_0.$$

Prove that the map $x_0 \mapsto t_+(x_0) \in (0, \infty]$ is lower semi-continuous. *Reminder:* A map g is called lower semi-continuous in x_0 if

$$\begin{array}{ll} \forall \varepsilon > 0 \;\; \exists \delta > 0 \;\; \forall x \quad (|x - x_0| < \delta \Rightarrow g(x) - g(x_0) > -\varepsilon)\,, & \text{in the case} \quad g(x_0) < \infty, \\ \forall \varepsilon > 0 \;\; \exists \delta > 0 \;\; \forall x \quad (|x - x_0| < \delta \Rightarrow g(x) > 1/\varepsilon)\,, & \text{in the case} \quad g(x_0) = \infty. \end{array}$$

Problem 30: Calculate the solutions of the following linear differential equations

$$\text{(i)} \qquad \dot{x} = \begin{pmatrix} -3 & 0 & 2 \\ -1 & -3 & 5 \\ -1 & 0 & 0 \end{pmatrix} x, \qquad x(0) = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

$$\text{(ii)} \qquad \dot{x} = \begin{pmatrix} 3 & 2 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{pmatrix} x, \qquad x(0) = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$$

Problem 31: Can matrices $A, B \in \mathbb{R}^{n \times n}$ fail to commute if $e^A = e^B = e^{A+B} = \mathrm{id}$? Prove your claim!

Problem 32: Consider the autonomous differential equation

$$\dot{x} = f(x), \quad x \in \mathbb{R}^N$$

with Lipschitz-continuous f. Suppose that all $x \in \mathbb{R}^N$ satisfy the inequality

$$(*) \qquad f(x)^T x \geq \|x\|^3_{\mathbb{R}^N}.$$

Prove: The maximal existence time $t_+(x_0)$ of the solution $x(t)$ is bounded, for every non-zero initial condition $x_0 \neq 0$.

Extra credit: Is $x = 0$ automatically an equilibrium if $(*)$ is satisfied?