Homework assignment

# Dynamical Systems

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http://dynamics.mi.fu-berlin.de/lectures/

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**Problem 37:** How many digits (in the decimal system) does the 1.000.000.000.001-st entry of the sequence $(1, 3, 8, 20, 48, 112, \ldots)$ have, i.e. $x_n = 4x_{n-1} - 4x_{n-2}$ with $x_0 = 1$ and $x_1 = 3$?

**Problem 38:** Consider the sequence

$$1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ \ldots,$$

i.e.

$$x_n \;=\; \frac{1}{2^n\sqrt{5}}\left(\left(1+\sqrt{5}\right)^n - \left(1-\sqrt{5}\right)^n\right).$$

(i) Give an interpretation of this sequence via iterations of a suitable linear map $A : \mathbb{R}^2 \to \mathbb{R}^2$. Determine the linear map and prove your claim.

(ii) For which nonzero initial conditions $(\tilde{x}_1, \tilde{x}_2) \in \mathbb{Z}^2$ to the above iteration does the quotient

$$r_n \;=\; \tilde{x}_{n+1}/\tilde{x}_n$$

**not** converge to the "golden ratio" $g = \frac{1}{2}(1+\sqrt{5})$ ?

**Problem 39:** Let $f$ be a differentiable vector field on $\mathbb{R}^3$.

Prove or disprove: A trajectory coincides with its $\omega$-limit set if, and only if, the trajectory is an equilibrium or a periodic orbit.

**Problem 40:** Consider the system of differential equations

$$\dot{x}_i \;=\; x_i\left((Ax)_i - x^{\mathrm{T}}Ax\right), \qquad i = 1,\dots,n,$$

for $x = (x_1,\dots,x_n) \in \mathbb{R}^n$ with $x_i \geq 0$ and

$$\sum_{i=1}^{n} x_i \;=\; 1.$$

Consider the case of the identity matrix, $A = \mathrm{id}$.

(i) Sketch the phase portraits for $n = 2, 3, 4$.

(ii) Describe the set of equilibria and the set of heteroclinic orbits for arbitrary $n$. In particular determine which equilibria are connected by heteroclinic orbits.

*Extra credit:* What happens for $\sum_{i=1}^{n} x_i \neq 1$ and/or $x_i < 0$ for some $i$ ?

*Hint:* Enumerate equilibria $x^*$ by subsets $M(x^*) = \{i|x_i^* \neq 0\} \subseteq \{1,\dots,n\}$.