Homework assignment

Dynamical Systems

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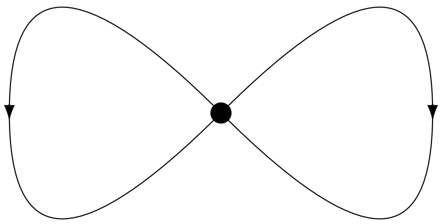
http://dynamics.mi.fu-berlin.de/lectures/

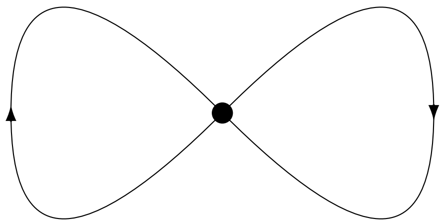
due date: Friday, July 2, 2010

Problem 41: Which of the following sets are possible ω-limits of a (single) trajectory of some planar flow? Which of the sets cannot occur as ω-limits (of a single trajectory)? Justify your claims, without providing explicit vector fields.

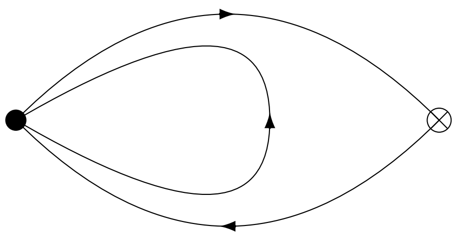
(i)



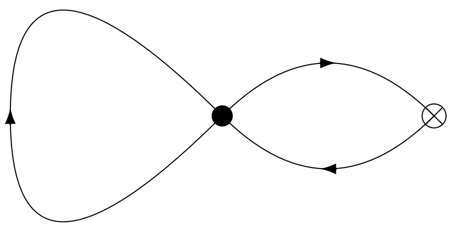
(ii)



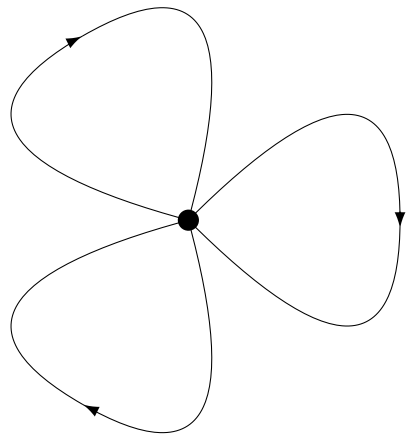
(iii)



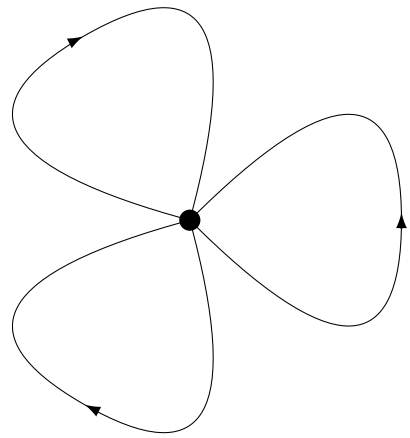
(iv)



(v)



(vi)



Discs ● denote equilibria (of any type) and crossed out circles $\otimes$ denote hyperbolic saddles.

Problem 42: Let $A \subseteq B \subseteq X = \mathbb{R}^N$ be sets and φ_t a flow on X. The set A is called *chain-recurrent* with respect to B if for every $y_0 \in A$ and every $\varepsilon > 0$, $T > 0$ there exists a positive number $n \in \mathbb{N}$, a sequence of times $t_0, \ldots, t_{n-1} \geq T$, and points $y_1, \ldots, y_{n-1} \in B$ such that

$$\operatorname{dist}(\varphi_{t_i}(y_i), y_{i+1}) < \varepsilon, \qquad i = 0, \ldots, n-1 \quad (\text{mod } n, \text{ i.e. } y_n := y_0).$$

The set A is called *recurrent*, if we can chose chains of length $n = 1$ for all points, i.e. if $y_0 \in \omega(y_0)$ for all $y_0 \in A$.

Prove: For any $x_0 \in X$, the ω-limit $\omega(x_0)$ is chain-recurrent with respect to X, but it is not necessarily recurrent.

Extra credit: Let the trajectory $\varphi_t(x_0)$ be bounded. Prove or disprove: The ω-limit $\omega(x_0)$ is chain-recurrent with respect to *itself*.

Problem 43: Choose at least one variant:

(A) In a bounded planar classroom, a student with a difficult question chases a professor. Both run at constant velocity 1 in arbitrary (and variable) direction. Can the student reach the professor?

(B) The athletic professor runs along a circle in $\mathbb{R}^2$, with constant velocity 1. The student starts at the center and keeps running towards the current position of the professor, at constant velocity $p < 1$. What is the student's ω-limit set? Is the ω-limit set stable? What happens for $p = 1$, $p > 1$?

Hint (B): Useful coordinates are the angle between origin and student, as seen from the professor, and the distance between student and professor.

Extra "credit": Chase the student.

Problem 44: Consider again the Feinberg result on reversible chemical networks of deficiency zero as discussed in class:

$$\dot{x} \;=\; \sum_{y \to y'} k_{y \to y'} x^y (y' - y),$$

with concentrations $x \in \mathbb{R}^n_+$, stoichiometric coefficients $y \in \mathbb{N}^n_0$, and reaction rates $k_{y \to y'}$. Given an initial value $x(0) \in \mathbb{R}^n_+$ with positive components, the stoichiometric subspace $(S + x(0)) \cap \mathbb{R}^n_+$,

$$S \;=\; \text{span}\, \{\, y' - y \mid k_{y \to y'} > 0 \,\}\,,$$

is invariant and contains a unique equilibrium x^*. Furthermore, there exists a strict Lyapunov function

$$V(x) \;=\; \sum_{i=1}^{n} x_i \,(\log x_i - \log x_i^* - 1)$$

on $(S + x(0)) \cap \mathbb{R}^n_+$.

Prove that these claims imply that the equilibrium x^* is (locally) asymptotically stable in $S + x(0)$.