Voluntary problems

Dynamical Systems

Bernold Fiedler, Stefan Liebscher

http://dynamics.mi.fu-berlin.de/lectures/

Problem Z1: Consider two linear dynamical systems in the plane,

$$(1) \qquad \dot{x} \;=\; Ax, \qquad\qquad (2) \qquad \dot{x} \;=\; Bx,$$

$x = (x_1, x_2) \in \mathbb{R}^2$, $A = (a_{ij})_{1\le i,j\le 2}$, $B = (b_{ij})_{1\le i,j\le 2}$. Combine both system to the piecewise linear system

$$(3) \qquad \dot{x} \;=\; \begin{cases} Ax & \text{for} \quad x_1x_2 > 0 \\ Bx & \text{for} \quad x_1x_2 < 0 \end{cases}$$

We call a function $x : (0,T) \to \mathbb{R}$ a solution of (3) to the initial condition $x(0)$ if

- x is Lipschitz-continuous and piecewise differentiable.
- x solves (3) at each point $x(t)$ of the set

$$\Omega \;=\; \{\, (x_1,x_2) \in \mathbb{R}^2 \,|\, x_1x_2 \neq 0 \,\} \cup \{\, (0,0) \,\}$$

- x intersects the complement of Ω only at discrete times, i.e. the set $\{t \in I \,|\, x(t) \notin \Omega\}$ is discrete in $\mathbb{R}$.

(i) Prove: If $a_{12}b_{12} > 0$ and $a_{21}b_{21} > 0$ then system (3) has a global and unique solution for any initial condition.

(ii) Prove or disprove: If, under the assumptions of (i), the origin is asymptotically stable in (1) as well as (2), i.e. if $\Re\mathrm{e}\,\mathrm{spec}\,A < 0$ and $\Re\mathrm{e}\,\mathrm{spec}\,B < 0$, then the origin is asymptotically stable in (3).

Problem Z2: Let $I \subset \mathbb{R}$ be an interval and $A \in C^1(I, \mathbb{R}^{n\times n})$.

Prove: If A and $\dot{A}$ commute, i.e. if $[A(t), \dot{A}(t)] := A(t)\dot{A}(t) - \dot{A}(t)A(t) = 0$ for all $t \in I$, then

$$\frac{\mathrm{d}}{\mathrm{d}t} e^{A(t)} \;=\; \dot{A}(t)\, e^{A(t)} \;=\; e^{A(t)}\, \dot{A}(t).$$

Problem Z3: Consider a continuous flow on X and a non-empty, compact, and invariant subset $M \subset X$.

Prove or disprove: M is stable if, and only if, every neighborhood of M contains a positively invariant neighborhood of M.

Recall: A neighborhood of a set A in Y is any set N which contains an open set U such that $\mathrm{clos}(A) \subseteq U \subseteq N \subseteq Y$.

Problem Z4: Consider the differential equation

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= (1-x^2-y^2)y - x.\end{aligned}$$

Prove: there exists a *unique* periodic orbit.

Problem Z5: Consider the prey-predator system

$$\begin{aligned}\dot{x} &= x(1-ax-y),\\ \dot{y} &= y(-c+x-by),\end{aligned}$$

with $(x,y) \in \mathbb{R}^2_+$, and parameters $a>0$, $b>0$, $c>0$, $ac<1$.

Prove:

(i) there exists a unique equilibrium (x_*, y_*);

(ii) $\omega((x_0,y_0)) = \{(x_*,y_*)\}$, for all initial conditions $x_0>0$, $y_0>0$.

Hint: Remember the case $a=b=0$, and compare with it.

Problem Z6: Prove or disprove: Every three-dimensional divergence-free differentiable vector field

$$f : \mathbb{R}^3 \to \mathbb{R}^3, \qquad \operatorname{div} f \equiv 0,$$

possesses a regular First Integral $I : \mathbb{R}^3 \to \mathbb{R}$.

Note: a First Integral is called regular if, and only if, its gradient vanishes only at zeros of the vector field, i.e

$$\forall\, x \in \mathbb{R}^3 \;\; \nabla I(x) = 0 \implies f(x) = 0.$$

Problem Z7: Can an unstable equilibrium position become stable upon linearization? Can it become asymptotically stable? Can an asymptotically stable equilibrium become unstable?

Problem Z8: Let $y(t)$ be a continuous function solving the integral equation

$$y(t) = \int_0^t y(s)^\alpha \sin(2010\, y(s))\, \mathrm{d}s,$$

for some fixed $\alpha \geq 0$ and all $t \in [0,1]$.

Prove: $y(t) \equiv 0$ is constant for all $t \in [0,1]$. For which $\alpha \in [-1,0)$ does the claim hold?

Problem Z9: [Arnol'd, (Russian) sample examination problems] It is known from experience that when light is refracted a the interface between media, the sines of the angles formed by the incident and refracted rays with the normal to the interface are inversely proportional to the indices of refraction of the media:

$$\frac{\sin \alpha_1}{\sin \alpha_2} = \frac{n_2}{n_1}.$$

Find the form of the light rays in the plane $(x, y) \in \mathbb{R}^2$ if the index of refraction is $n = n(y)$. Study the case $n(y) = 1/y$.

Remark: The half plane $\{y > 0\}$ with the index of refraction $n(y) = 1/y$ gives a model of Lobachevskian geometry.

Problem Z10: [Arnol'd, (Russian) sample examination problems] Draw the rays emanating in different directions from the origin in a plane with index of refraction $n = n(y) = y^4 - y^2 + 1$.

Remark: This explains the formation of a mirage: the air over a desert has its maximum of refraction in a certain finite height, due to more rarefied air in higher and lower layers.

Acoustic channels in the ocean are a similar phenomenon: the maximum of rarefaction is found at a depth of 500m-1000m.